p-ADIC EISENSTEIN-KRONECKER FUNCTIONS AND THE ELLIPTIC POLYLOGARITHM FOR CM ELLIPTIC CURVES

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Abstract. In this paper, we construct p-adic analogues of the Kronecker double series, which we call the Eisenstein-Kronecker series, as Coleman functions on an elliptic curve with complex multiplication. We then show that the periods of the specialization of the p-adic elliptic polylogarithm sheaf to arbitrary non-zero points of the elliptic curve may be expressed using these functions.

1. Introduction

1.1. **Introduction.** Let Γ be a lattice in $\mathbb{C}$, and suppose A is the fundamental area of Γ divided by $\boldsymbol{\pi} := 3.1415....$ The classical Eisenstein-Kronecker functions, more commonly known as the Kronecker double series, are defined to be the series

$$E_{m,n}(z) := \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{\chi_z(\gamma)}{\overline{\gamma}^m \gamma^n} \tag{1}$$

for integers m and n, where χ_z is the character $\chi_z(\gamma) = \exp((\gamma \overline{z} - \overline{\gamma} z)/A)$ for any complex number z. The series converge only for $m+n > 2$, but one may give meaning to the above functions for general integers m and n by analytic continuation. In the context of polylogarithms, these functions are elliptic analogues of the Bloch-Wigner-Zagier polylogarithm function (see [L] p.282 Remark). In the paper [Col1], Coleman defined the p-adic analogue of the classical polylogarithm function as a Coleman function, which is a class of p-adic analytic functions generalizing the rigid analytic functions. The purpose of this paper is to define a p-adic analogue of the Eisenstein-Kronecker function as a Coleman function, when the complex torus $\mathbb{C}/\Gamma$ has a model as an elliptic curve with complex multiplication defined over the field of complex multiplication, with good reduction at the primes above $p \geq 5$. Our main result states that the periods of the p-adic elliptic polylogarithm sheaf may be expressed using these functions.

In a fundamental paper, Beilinson and Levin [BL] defined the elliptic polylogarithm sheaf as a certain variation of mixed Hodge structures on an elliptic curve minus the identity, extending previous constructions of

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Beilinson and Deligne for the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The construction is motivic, in a sense that it works for any reasonable cohomology theory, including absolute Hodge cohomology, étale cohomology and rigid syntomic cohomology. In [Ba3] and [BKT1], we applied this construction to the rigid syntomic case, and constructed the p-adic elliptic polylogarithm sheaf as a filtered overconvergent F-isocrystal, which is a p-adic analogue of a variation of mixed Hodge structures, on an elliptic curve minus the identity, when the elliptic curve has complex multiplication and good reduction at $p \geq 5$. We proved in particular that the specialization of this sheaf to *torsion points of order prime to* p is related to special values of p-adic distributions interpolating Eisenstein-Kronecker numbers.

The main result of this paper is the calculation of the specializations of the p-adic elliptic polylogarithm sheaf to *arbitrary non-zero points* of the elliptic curve. This is an elliptic analogue of the results of [Som] and [BdJ]. We use ideas from [Ba4] to deal with technical difficulties which arise from the fact that such points are in general defined over ramified extensions of the base field. We prove in particular that the specializations may be explicitly expressed in terms of special values of the p-adic Eisenstein-Kronecker functions. Since the construction of the elliptic polylogarithm is motivic, our result amounts to the explicit calculation of the image of motivic elements with respect to the syntomic regulator, modulo some expected compatibility (see the introduction of [BKT1] for this point).

The interest in looking at non-torsion points comes from the elliptic analogue of the Zagier conjecture [W2]. The p-adic Eisenstein-Kronecker functions which we have defined should play a future role in formulating and investigating p-adic analogues of this conjecture.

Contents

2. Eisenstein-Kronecker series and the Kronecker theta function

2.1. **Eisenstein-Kronecker series.** In this section, we review the definition of the Eisenstein-Kronecker series. We first review the definition of the Eisenstein-Kronecker-Lerch series.

We fix a lattice $\Gamma \subset \mathbb{C}$, and we let $\chi_w(z) := \exp((z\overline{w} - w\overline{z})/A)$ for any z, $w \in \mathbb{C}$, where A is the fundamental area of Γ divided by $\boldsymbol{\pi}$. We define

the Eisenstein-Kronecker-Lerch function following Weil ([We] VIII §12) as follows.

Definition 2.1. Let a be an integer, and let z_0, $w_0 \in \mathbb{C}$. We define the Eisenstein-Kronecker-Lerch series by

$$K_a^*(z_0, w_0, s) = \sum_{\gamma \in \Gamma \setminus \{-z_0\}} \frac{(\overline{z}_0 + \overline{\gamma})^a}{|z_0 + \gamma|^{2s}} \chi_{w_0}(\gamma).$$

The above series converges for $\mathrm{Re}(s) > a/2 + 1$, and extends to a meromorphic function on the whole complex plane by analytic continuation. The following result was proved by Weil [We] VIII §13. See also [BKT1] Proposition 2.2 for details concerning the case $a \leq 0$.

Proposition 2.2. *Let a be an integer.*

(1) *The function $K_a^*(z_0, w_0, s)$ for s continues meromorphically to a function on the whole s plane, with a simple pole only at $s = 1$ if $a = 0$ and $w_0 \in \Gamma$.*
(2) *The function $K_a^*(z_0, w_0, s)$ satisfies the functional equation*

$$\Gamma(s) K_a^*(z_0, w_0, s) = A^{a+1-2s} \Gamma(a + 1 - s) K_a^*(w_0, z_0, a + 1 - s) \chi_{w_0}(z_0).$$

The Eisenstein-Kronecker numbers are defined as the special values of Eisenstein-Kronecker-Lerch series (see also [BK1] Definition 1.5 and [BKT1] Definition 2.3).

Definition 2.3. Let z_0, $w_0 \in \mathbb{C}$, and let a and b be integers such that $(a, b) \neq (1, -1)$ if $w_0 \in \Gamma$. The Eisenstein-Kronecker number $e_{a,b}^*(z_0, w_0)$ is defined by $e_{a,b}^*(z_0, w_0) := K_{a+b}^*(z_0, w_0, b)$. As in [BKT1] Definition 2.3, we let

$$e_{a,b}^*(z_0) := e_{a,b}^*(0, z_0) = K_{a+b}^*(0, z_0, b)$$

for $z_0 \in \mathbb{C}$ such that $z_0 \notin \Gamma$ if $(a, b) \neq (-1, 1)$.

We define the Eisenstein-Kronecker function, referred to more commonly as the Kronecker double series, as follows.

Definition 2.4. For any integer m, n, we define the Eisenstein-Kronecker function $E_{m,n}(z)$ to be the $\mathscr{C}^\infty$-function on $\mathbb{C} \setminus \Gamma$ defined by

$$E_{m,n}(z) := K_{n-m}^*(0, z, n).$$

The Eisenstein-Kronecker function coincides with the power series expansion (1) of the introduction if $m + n > 2$. This function is known to satisfy the differential equations

$$\partial_z E_{m,n}(z) = -E_{m-1,n}(z)/A, \qquad \partial_{\overline{z}} E_{m,n}(z) = E_{m,n-1}(z)/A.$$

Remark 2.5. In [BKT1], we used a different normalization. The Eisenstein-Kronecker function $E_{m,n}(z)$ in [BKT1] is $A^{m+n-1} E_{m,n}(z)$ in our notation. See [BKT1] Definition 2.6 for details.

Note that by definition, we have

$$E_{m,n}(z_0) = e^*_{-m,n}(z_0)$$

for any $z_0 \in \mathbb{C} \setminus \Gamma$.

2.2. **Kronecker theta function.** We next review the definition and basic properties of the Kronecker theta function. Let $\Gamma \subset \mathbb{C}$ be a lattice, and denote by $\theta(z)$ the reduced theta function associated to the divisor $[0]$ of $\mathbb{C}/\Gamma$, normalized so that $\theta'(0) = 1$. This function is given explicitly as

$$\theta(z) = \exp(-e^*_{0,2} z^2/2)\sigma(z),$$

where $\sigma(z)$ is the Weierstrass sigma function

$$\sigma(z) := z \prod_{\gamma \in \Gamma \setminus \{0\}} \left(1 - \frac{z}{\gamma}\right) \exp\left[\frac{z}{\gamma} + \frac{z^2}{2\gamma^2}\right]$$

associated to the lattice Γ, and $e^*_{0,2} := \lim_{s \to 0} \sum_{\gamma \in \Gamma \setminus \{0\}} \gamma^{-2} |\gamma|^{-2s}$ which coincides with the Eisenstein-Kronecker number $e^*_{0,2}(0)$. We define the Kronecker theta function as follows.

Definition 2.6 (Kronecker theta function)**.** We let

$$\Theta(z, w) := \theta(z + w)/\theta(z)\theta(w).$$

The relation of this function to the two-variable Jacobi theta function $F_\tau(z, w)$ of Zagier [Zag] is given by $\Theta(z, w) = \exp(zw/A) F_\tau(z, w)$. For any z_0, $w_0 \in \mathbb{C}$, we let

$$\Theta_{z_0,w_0}(z, w) := \exp\left[-\frac{z_0 \overline{w}_0}{A}\right] \exp\left[-\frac{z\overline{w}_0 + w\overline{z}_0}{A}\right] \Theta(z + z_0, w + w_0).$$

This function is known to be the generating function of Eisenstein-Kronecker numbers as follows (see [BK1] §1.14 Theorem 1.17).

Theorem 2.7. *We have*

$$\Theta_{z_0,w_0}(z, w) = \chi_{z_0}(w_0) \frac{\delta_{z_0}}{z} + \frac{\delta_{w_0}}{w} + \sum_{a,b \geq 0} (-1)^{a+b} \frac{e^*_{a,b+1}(z_0, w_0)}{a! A^a} z^b w^a$$

in a neighborhood of the origin, where $\delta_x = 1$ if $x \in \Gamma$ and is zero otherwise.

We define the function $F_{z_0,b}(z)$ as in [BKT1] Definition 4.2 as follows.

Definition 2.8. We let $F_{z_0,b}(z)$ be the functions such that

$$\Theta_{z_0,0}(z, w) = \sum_{b \geq 0} F_{z_0,b}(z) w^{b-1}.$$

The choice of z_0 in the definition of $F_{z_0,b}(z)$ depends only the class of z_0 modulo Γ. When $z_0 = 0$, we let $F_b(z) := F_{0,b}(z)$. Explicit calculations show that we have $F_0(z) = 1$ and $F_1(z) = \theta'(z)/\theta(z) = \zeta(z) - e^*_{0,2} z$. Note that we have by definition $F_{z_0,1}(z) = F_1(z + z_0) - \overline{z}_0/A$. We will later show that

in the p-adic case, $F_{z_0,b}$ for various z_0 paste together to form a Coleman function.

The formula for $\Theta_{z_0,w_0}(z,w)$ as the generating function for Eisenstein-Kronecker numbers gives the following.

Proposition 2.9 (Generating function)**.** *For any $b \geq 0$, the Laurent expansion of $F_{z_0,b}(z)$ at 0 is given by*

$$F_{z_0,b}(z) = \frac{\delta_{z_0,b}}{z} + \sum_{a \geq 0} (-1)^{a+b-1} \frac{e^*_{a,b}(z_0)}{a! A^a} z^a,$$

where $\delta_{x,b} = 1$ if $b = 1$ and $x \in \Gamma$ and is zero otherwise.

Proof. See for example [BKT1] Corollary 4.3. □

We next define an auxiliary function $L_n(z)$ used to define the connection of the elliptic polylogarithm sheaf.

Definition 2.10. Let $\Xi(z,w) := \exp(-F_1(z)w)\Theta(z,w)$. We define the connection function $L_n(z)$ by the formula

$$\Xi(z,w) := \sum_{n \geq 0} L_n(z) w^{n-1}.$$

Remark 2.11. Explicit calculation shows that the connection function for small n is given by $L_0(z) = 1$, $L_1(z) = 0$ and $L_2(z) = -\frac{1}{2}\wp(z)$.

The function $L_n(z)$ is a periodic function on $\mathbb{C}/\Gamma$, holomorphic outside Γ. The relation between $F_{z_0,b}(z)$ and $L_n(z)$ is given by the formula

$$F_{z_0,b}(z) = \sum_{n=0}^{b} \frac{F_{z_0,1}(z)^{b-n}}{(b-n)!} L_n(z+z_0). \tag{2}$$

Next, we assume that our complex torus has an algebraic model. Let F be a number field with a fixed embedding $F \hookrightarrow \mathbb{C}$, and assume that we have an elliptic curve E over F defined by the Weierstrass equation

$$E : y^2 = 4x^3 - g_2 x - g_3. \tag{3}$$

We let Γ be the period lattice of E with respect to the invariant differential $\omega = dx/y$. We have a complex uniformization $\xi : \mathbb{C}/\Gamma \cong E(\mathbb{C})$ such that dz corresponds to ω. Then the connection function $L_n(z)$ is algebraic in the following sense.

Proposition 2.12. *The functions $L_n(z)$ correspond through the uniformization ξ to rational functions L_n on E defined over F.*

Proof. See [BKT1] Proposition 1.6. □

Assume now that E has complex multiplication by the ring of integers $\mathcal{O}_{\boldsymbol{K}}$ of an imaginary quadratic field $\boldsymbol{K}$. In this case, the function $\Theta_{z_0,w_0}(z,w)$ satisfies the following algebraicity result.

Theorem 2.13 (Damerell)**.** *Suppose z_0, w_0 correspond to torsion points in $\mathbb{C}/\Gamma \cong E(\mathbb{C})$. Then we have*

$$\Theta_{z_0,w_0}(z,w) - \chi_{z_0}(w_0) z^{-1}\delta_{z_0} - w^{-1}\delta_{w_0} \in \overline{\mathbb{Q}}[[z,w]],$$

where $\delta_x = 1$ if $x \in \Gamma$ and $\delta_x = 0$ otherwise.

Proof. This is a reformulation of the classical theorem of Damerell. See [BK1] Corollary 2.4 for a proof. □

Corollary 2.14. *Suppose z_0 corresponds to a torsion point in $\mathbb{C}/\Gamma \cong E(\mathbb{C})$. Then we have*

$$F_{z_0,b}(z) - \delta_{z_0,b} z^{-1} \in \overline{\mathbb{Q}}[[z]],$$

where $\delta_{x,b} = 1$ if $b = 1$ and $x \in \Gamma$, and $\delta_{x,b} = 0$ otherwise.

We fix an isomorphism $[\,] : \mathcal{O}_{\boldsymbol{K}} \cong \mathrm{End}_{\overline{F}}(E)$ so that $\alpha \in \mathcal{O}_{\boldsymbol{K}}$ acts as $[\alpha]^*\omega = \alpha\omega$ on the invariant differential $\omega = dx/y$. For any non-zero $\alpha \in \mathcal{O}_{\boldsymbol{K}}$, we denote by $E[\alpha]$ the subgroup of $E(\overline{\mathbb{Q}})$ annihilated by α. The function $F_{z_0,b}(z)$ is known to satisfy the following distribution relation with respect to $E[\alpha]$.

Proposition 2.15 (Distribution Relation)**.** *The function $F_{z_0,b}(z)$ satisfies the relation*

$$\sum_{z_\alpha \in E[\alpha]} F_{z_0+z_\alpha,b}(z) = \alpha\overline{\alpha}^{1-b} F_{\alpha z_0,b}(\alpha z)$$

for any non-zero $\alpha \in \mathcal{O}_{\boldsymbol{K}}$.

Proof. By [BK1] Proposition 1.16, noting that

$$\Theta_{\alpha z_0,0}(\alpha z, w/\overline{\alpha}; \Gamma) = \overline{\alpha}\Theta_{N(\alpha)z_0,0}(N(\alpha)z, w; \overline{\alpha}\Gamma),$$

we see that the Kronecker theta function $\Theta_{z_0,0}(z,w)$ satisfies the distribution relation

$$\sum_{z_\alpha \in E[\alpha]} \Theta_{z_0+z_\alpha,0}(z,w) = \alpha\Theta_{\alpha z_0,0}(\alpha z, w/\overline{\alpha})$$

for any non-zero $\alpha \in \mathcal{O}_{\boldsymbol{K}}$. Our assertion follows from the definition of $F_{z_0,b}(z)$. □

2.3. **The p-adic properties.** We now review the p-adic properties of the function $F_{z_0,b}(z)$. We let E be an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_{\boldsymbol{K}}$ of an imaginary quadratic field $\boldsymbol{K}$. We assume in addition that E is defined over $\boldsymbol{K}$ and has good reduction at the primes above $p \geq 5$. This implies that p does not ramify in $\mathcal{O}_{\boldsymbol{K}}$. We fix a Weierstrass model of E over $\mathcal{O}_{\boldsymbol{K}}$ with good reduction above p, and by abuse of notations, we denote again by E this model defined over $\mathcal{O}_{\boldsymbol{K}}$. Let $t = -2x/y$ be the formal parameter of E at the origin, and we denote by $\widehat{E}$ the formal group of E with respect to the parameter t. We denote by $\lambda(t)$ the formal logarithm.

Let $z_0 \in E(\overline{\mathbb{Q}})_{\mathrm{tor}}$, and we denote by $\widehat{F}_{z_0,b}(t) := F_{z_0,b}(z)|_{z=\lambda(t)}$ the formal composition of the Laurent expansion of $F_{z_0,b}(z)$ at the origin with the formal power series $z = \lambda(t)$. Note that by definition, we have

$$\widehat{F}_{z_0,b}(t) = \sum_{n=0}^{b} \frac{\widehat{F}_{z_0,1}(t)^{b-n}}{(b-n)!} \widehat{L}_{z_0,n}(t), \tag{4}$$

where $\widehat{L}_{z_0,n}(t) := L_n(z+z_0)|_{z=\lambda(t)}$.

Let $\psi := \psi_{E/\boldsymbol{K}}$ be the Hecke character of $\boldsymbol{K}$ associated to E. We let $\mathfrak{p}$ be a prime in $\mathcal{O}_{\boldsymbol{K}}$ lifting p, and let $\pi := \psi(\mathfrak{p})$. Note that if p is ordinary, then π is an element such that $p = \pi\pi^*$ with π^* a unit in $\boldsymbol{K}_{\mathfrak{p}}$, and $\pi = -p$ if p is supersingular. We fix an embedding $\overline{\boldsymbol{K}} \hookrightarrow \mathbb{C}_p$ such that π maps to an element of absolute value < 1 in $\mathbb{C}_p$. By Corollary 2.14, we may view $\widehat{F}_{z_0,b}(t)$ as a power series with coefficients in $\mathbb{C}_p$ through this embedding. We next review the p-adic properties of the power series $\widehat{F}_{z_0,b}(t)$ through this embedding.

Proposition 2.16. *Let z_0 be a torsion point in $E(\overline{\mathbb{Q}})$ of order prime to $\mathfrak{p}$. Then the radius of convergence of the holomorphic part*

$$\widehat{F}_{z_0,b}(t) - \delta_{z_0,b} t^{-1} \in \mathbb{C}_p[[t]]$$

of $\widehat{F}_{z_0,b}(t)$ is one. In other words, this power series defines a function on $\mathbb{B}^- := \{t \in \mathbb{C}_p \mid |t| < 1\}$ if $b \neq 1$ or $z_0 \neq 0$, and $\widehat{F}_1(t) := \widehat{F}_{0,1}(t)$ defines a function on $\mathbb{A}(0) := \{t \in \mathbb{C}_p \mid 0 < |t| < 1\}$.

Proof. See [BKT1] Proposition 4.7 for the proof. □

In the next section, we will prove that the power series $\widehat{F}_{z_0,b}(t)$ paste together to form a Coleman function on the elliptic curve minus the identity. We next review a formula for translation by π^n-torsion points.

Lemma 2.17 (Translation)**.** *Suppose z_0 is a torsion point in $E(\overline{\mathbb{Q}})$ of order prime to $\mathfrak{p}$. Then we have*

$$\widehat{F}_{z_0,b}(t \oplus t_n) = \widehat{F}_{z_0+z_n,b}(t),$$

where $t_n \in \widehat{E}[\pi^n]$, and z_n is the image of t_n through the inclusion $\widehat{E}(\mathfrak{m}_p)_{\mathrm{tor}} \subset E(\overline{\mathbb{Q}})_{\mathrm{tor}} \subset \mathbb{C}/\Gamma$, where $\mathfrak{m}_p$ is the prime ideal of $\mathcal{O}_{\mathbb{C}_p}$.

Proof. See [BKT1] Lemma 4.13 for the proof. □

The above lemma gives the following corollary.

Corollary 2.18 (Generating function)**.** *Suppose z_0 is a non-zero torsion point in $E(\overline{\mathbb{Q}})$ of order prime to $\mathfrak{p}$. Then for any integer $a \geq 0$, we have*

$$(\partial^a_{t,\log} \widehat{F}_{z_0,b}(t))|_{t=t_n} = (-1)^{a+b-1} e^*_{a,b}(z_0+z_n)/A^a,$$

where $\partial_{t,\log}$ is the differential operator $\lambda'(t)^{-1}\partial_t$.

Proof. Note that if we let $z = \lambda(t)$, then we have $\partial_z = \partial_{t,\log}$. Our assertion follows from Lemma 2.17 and the generating function property of $F_{z_0,b}$ given in Proposition 2.9. $\square$

We let $F_1^{(p)}(z)$ be the function

$$F_1^{(p)}(z) := F_1(z) - \overline{\pi}^{-1} F_1(\pi z), \tag{5}$$

which is an elliptic function corresponding to a rational function defined over $\boldsymbol{K}$. Then we have $F_1^{(p)}(z + z_0) = F_{z_0,1}(z) - \overline{\pi}^{-1} F_1(\pi z)$, hence

$$F_1^{(p)}(z + z_0)|_{z=\lambda(t)} = \widehat{F}_{z_0,1}(t) - \overline{\pi}^{-1} F_{z_0,1}([\pi]t).$$

3. p-adic Eisenstein-Kronecker function

3.1. **Review of Coleman integration.** We first review the theory of Coleman integration for curves, following the description of Besser ([Bes2], [Bes3]), using notations coming from rigid cohomology.

Let K be a complete subfield of $\mathbb{C}_p$, with ring of integers $\mathcal{O}_K$ and residue field k. Let X be a smooth irreducible projective scheme over $\mathcal{O}_K$ of relative dimension one. Let $U \subset X$ be an open affine and smooth subscheme of X such that the complement $X \setminus U$ is horizontal over $\mathcal{O}_K$.

Denote by $X^{\mathrm{an}}_{\mathbb{C}_p}$ be the rigid analytic space associated to the scheme $X_{\mathbb{C}_p}$. Its points consist of the points of $X(\mathbb{C}_p)$. We have the specialization morphism

$$\mathrm{sp} : X^{\mathrm{an}}_{\mathbb{C}_p} \to X_{\overline{\mathbb{F}}_p},$$

and we denote the inverse image of a point $x \in X(\overline{\mathbb{F}}_p)$ by $]x[$, which we call the residue disc. The set $]x[\subset X^{\mathrm{an}}_{\mathbb{C}_p}$ is an admissible open set of $X^{\mathrm{an}}_{\mathbb{C}_p}$. We have a set-theoretic decomposition

$$X^{\mathrm{an}}_{\mathbb{C}_p} = \coprod_{x \in X(\overline{\mathbb{F}}_p)}]x[,$$

which is *not* an admissible covering for the rigid topology. Since X is smooth, each residue disc is isomorphic to the open disc

$$\mathbb{B}^- := \{t \in \mathbb{C}_p \mid |t| < 1\}$$

through a choice of local parameter t_x of X at x. Let r be such that $0 < r < 1$. We define $\mathbb{A}(r)$ be the annulus $\mathbb{A}(r) := \{t \in \mathbb{C}_p \mid r < |t| < 1\}$, which is an admissible open of $\mathbb{B}^-$.

Definition 3.1. We let $A(]x[)$ be the ring of functions defined by

$$A(]x[) := \Gamma(\mathbb{B}^-, \mathcal{O}_{\mathbb{B}^-}) \qquad x \in U(\overline{\mathbb{F}}_p)$$

$$A(]x[) := \bigcup_{0<r<1} \Gamma(\mathbb{A}(r), \mathcal{O}_{\mathbb{A}(r)}) \qquad \text{otherwise,}$$

where we have identified $]x[$ with the rigid analytic space $\mathbb{B}^-$ through a choice of parameter t_x.

Note that $A(]x[)$ is simply the ring consisting of formal power series $f(t_x) = \sum_{n\geq 0} a_n t_x^n$ which converges on $\mathbb{B}^-$ if $x \in U(\overline{\mathbb{F}}_p)$, and formal power series $f(t_x) = \sum_{n=-\infty}^{\infty} a_n t_x^n$ which converges on $\mathbb{A}(r)$ for some $r < 1$ if $x \in (X \setminus U)(\overline{\mathbb{F}}_p)$. This definition is independent of the choice of the parameter t_x.

Definition 3.2. A branch of the p-adic logarithm is any locally analytic homomorphism $\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p$, defined by the power series

$$\log_p(x) = -\sum_{n>0} \frac{(1-x)^n}{n}$$

for x in a neighborhood of 1. It is characterized by the value $\log_p(p)$.

Suppose a branch of the p-adic logarithm has been chosen. One defines $A_{\log}(]x[)$ to be $A(]x[)$ if $x \in U(\overline{\mathbb{F}}_p)$ and to be the polynomial ring in the function $\log(t_x)$ over $A(]x[)$ if $x \in (X\setminus U)(\overline{\mathbb{F}}_p)$. This definition is independent of the choice of the local parameter t_x. Set $\Omega^1_{\log}(]x[) := A_{\log}(]x[)dt_x$. Then one defines the ring of locally analytic functions and one forms on U by

$$A_{\mathrm{loc}}(U) := \prod_{x \in X(\overline{\mathbb{F}}_p)} A_{\log}(]x[), \qquad \Omega^1_{\mathrm{loc}}(U) := \prod_{x \in X(\overline{\mathbb{F}}_p)} \Omega^1_{\log}(]x[).$$

We have a differential $d : A_{\mathrm{loc}}(U) \to \Omega^1_{\mathrm{loc}}(U)$, which is surjective.

Suppose $k = \mathbb{F}_q$ for $q = p^h$, and let $\mathcal{X}$ and $\mathcal{U}$ be the formal completions of X and U with respect to the special fiber. For simplicity, we assume that there exists a Frobenius morphism $\phi : \mathcal{X} \to \mathcal{X}$, which is a $\mathcal{O}_K$-linear morphism lifting the r-th power Fr^h of the absolute Frobenius Fr on $X_k := X \otimes k$ and such that $\phi(\mathcal{U}) \subset \mathcal{U}$. Then this map induces a $\mathbb{C}_p$-linear morphism $\phi : X^{\mathrm{an}}_{\mathbb{C}_p} \to X^{\mathrm{an}}_{\mathbb{C}_p}$ by extension of scalars.

Coleman constructs a certain subring $M(U)$ of $A_{\mathrm{loc}}(U)$, which we call the space of Coleman functions on U, equipped with an integration map. If we denote by $M(U)/\mathbb{C}_p$ the ring $M(U)$ modulo addition by constants, then the integration map is a vector space map $\int : M(U) \otimes_{A(U)} \Omega^1(U) \to M(U)/\mathbb{C}_p$ characterized by the following three properties.

(1) We have $d(\int \omega) = \omega$ (primitive function).
(2) We have $\int(\phi^*\omega) = \phi^*(\int \omega)$ (Frobenius invariance).
(3) If $g \in M(U)$, then $\int dg = g + \text{constant}$.

$M(U)$ is defined so that it contains rational functions on X which are regular on U, as well as overconvergent functions on $\mathcal{U}_{\mathbb{C}_p} \subset X^{\mathrm{an}}_{\mathbb{C}_p}$, where $\mathcal{U}_{\mathbb{C}_p}$ is the rigid analytic space associated to $\mathcal{U}$.

The construction of Coleman functions gives the following lemma.

Lemma 3.3. *Suppose f is a function in $A_{\mathrm{loc}}(U)$, and suppose $P(x)$ is a polynomial in $\mathbb{C}_p$ whose roots do not contain the roots of unity. If we have $df \in M(U) \otimes \Omega^1_{\mathrm{loc}}(U)$ and $P(\phi^*)f \in M(U)$, then we have $f \in M(U)$.*

3.2. $F_{z_0,b}(z)$ **as a Coleman function.** We will next show that the functions $F_{z_0,b}(z)$ defined in the previous section defines a Coleman function on our elliptic curve. Let the notations be as in §2.3. In particular, we let E be the model over $\mathcal{O}_{\boldsymbol{K}}$ of our CM elliptic curve, with good reduction at the primes above p. Let K be a finite extension of $\boldsymbol{K}_{\mathfrak{p}}$ in $\mathbb{C}_p$, and by abuse of notations, we denote again by E the extension of E to the ring of integers $\mathcal{O}_K$ of K. Then for $\pi := \psi(\mathfrak{p})$, multiplication by $[\pi]$ induces a Frobenius $\phi : E \to E$. We denote by $E(\mathbb{C}_p) := E^{\mathrm{an}}_{\mathbb{C}_p}$ the extension to $\mathbb{C}_p$ of the rigid analytic space E^{an}_K. From now until the end of this paper, z will denote a variable on $E(\mathbb{C}_p)$.

Each residue disc of $E(\mathbb{C}_p)$ contains the Teichmüller representative, which is the unique element in the residue disc fixed by a certain power of the chosen Frobenius. For our choice of the Frobenius ϕ, the Teichmüller representative is a torsion point z_0 of order prime to $\mathfrak{p}$. If we let $t = -2x/y$, then this gives a local parameter of E at the origin. Then the open disc $\{t \in \mathbb{C}_p \mid |t| < 1\}$ represents the residue disc $]0[\subset E(\mathbb{C}_p)$ containing the identity of E. If we let z_0 be a torsion point of $E(\mathbb{C}_p)$ of order prime to $\mathfrak{p}$ and $]z_0[:= \tau_{z_0}(]0[)$ for the translation $\tau_{z_0} : E \to E$ defined by $\tau_{z_0}(z) := z+z_0$, then $]z_0[$ is precisely the residue disc containing z_0.

We first start by investigating the function $F_{z_0,1}(z)$. We let $U := E \setminus [0]$. Note that by Proposition 2.16, the power series $\widehat{F}_{z_0,b}(t)$ defines a function in $A(]z_0[)$ through the identification $]z_0[\cong \mathbb{B}^-$.

Lemma 3.4. *We let F_1^{col} be the function in $A_{\mathrm{loc}}(U)$ defined by*

$$F_1^{\mathrm{col}}|_{]z_0[} := \widehat{F}_{z_0,1}(t) \ \in \ A(]z_0[) \subset A_{\log}(]z_0[)$$

on each residue disc $]z_0[$, where z_0 is a torsion point in $E(\overline{\mathbb{Q}})$ of order prime to $\mathfrak{p}$. Then F_1^{col} is a Coleman function on U.

Proof. The differential form $dF_1 = \eta + e^*_{0,2}\omega$ is known to be a meromorphic differential form of the second kind on U defined over $\mathcal{O}_{\boldsymbol{K}}$. Since $dF_{z_0,1} = \tau^*_{z_0} dF_1$, we have $dF_1^{\mathrm{col}} = dF_1$. For any Coleman function $f(z)$ on U, denote by $f(\pi z)$ the Coleman function $[\pi]^* f(z)$. Consider the function

$$\widetilde{F}_1^{(p)}(z) := \left(1 - \frac{\phi^*}{\overline{\pi}}\right) F_1^{\mathrm{col}}(z) = F_1^{\mathrm{col}}(z) - \frac{1}{\overline{\pi}} F_1^{\mathrm{col}}(\pi z).$$

For any $z \in]z_0[$, we have $\widetilde{z} := \pi z \in]\pi z_0[$. Hence from definition and the fact that $[\pi] \circ \tau_{z_0} = \tau_{\pi z_0} \circ [\pi]$, we have

$$F_1^{\mathrm{col}}(\pi z)|_{]z_0[} = F_1^{\mathrm{col}}(\widetilde{z})|_{]\pi z_0[} = \widehat{F}_{\pi z_0,1}(\widetilde{t}) = \widehat{F}_{\pi z_0,1}([\pi] t),$$

where t and $\widetilde{t}$ are element in $\mathbb{B}^-$ corresponding to z and $\widetilde{z}$. This shows that

$$\widetilde{F}_1^{(p)}|_{]z_0[} = F_1^{(p)}|_{]z_0[}$$

where $F_1^{(p)}$ is the rational function on E defined in (5). Then Lemma 3.3 shows that F_1^{col} is a Coleman function, uniquely characterized by the property that $dF_1^{\mathrm{col}} = dF_1$ and $(1 - \phi^*/\overline{\pi})\, F_1^{\mathrm{col}} = F_1^{(p)}$. $\square$

Note that L_n is a rational function on E with poles only at $[0]$ in E, hence is in particular a Coleman function on U. The set of Coleman functions is a ring, and we define F_b^{col} as follows.

Definition 3.5. We let F_1^{col} to be the Coleman function of Lemma 3.4. For any integer $b > 1$, we define F_b^{col} to be the Coleman function

$$F_b^{\mathrm{col}} := \sum_{n=0}^{b} \frac{(F_1^{\mathrm{col}})^{b-n}}{(b-n)!} L_n \tag{6}$$

on U.

The functions F_b^{col} interpolate the power series $F_{z_0,b}(z)$ of the previous section.

Proposition 3.6 (Interpolation)**.** *The function F_b^{col} on the residue disc $]z_0[$ is given by*

$$F_b^{\mathrm{col}}|_{]z_0[} = \widehat{F}_{z_0,b}(t) \in A_{\log}(]z_0[).$$

Proof. The case for $b = 1$ follows from the definition of F_1^{col}. The case for $b > 1$ follows from this case, noting that $L_n|_{]z_0[} = \widehat{L}_{z_0,n}(t)$ and comparing the definitions of (4) and (6). □

Proposition 3.7 (Distribution Relation)**.** *For any non-zero $\alpha \in \mathcal{O}_{\boldsymbol{K}}$, the function F_b^{col} satisfies the distribution relation*

$$\sum_{z_\alpha \in E[\alpha]} F_b^{\mathrm{col}}(z + z_\alpha) = \alpha\overline{\alpha}^{1-b} F_b^{\mathrm{col}}(\alpha z).$$

Proof. We write $\alpha = \pi^n \alpha_0$, where $\alpha_0 \in \mathcal{O}_{\boldsymbol{K}}$ is prime to π. For any $z_\alpha \in E[\alpha]$, we write $z_\alpha = z_{\alpha_0} + z_n$, where $z_{\alpha_0} \in E[\alpha_0]$ and $z_n \in E[\pi^n]$. Suppose z is a point in the residue disc $]z_0[$. Then $z + z_\alpha$ is in the residue disc $]z_0 + z_{\alpha_0}[$, and αz is in $]\alpha z_0[$. Denote by t the element in $\mathbb{B}^-$ corresponding to z through the isomorphism $]z_0[\cong \mathbb{B}^-$, and by t_n the element corresponding to z_n. Then Proposition 3.6 shows that we have

$$F_b^{\mathrm{col}}(\alpha z) = \widehat{F}_{\alpha z_0,b}([\alpha]t),$$
$$F_b^{\mathrm{col}}(z + z_\alpha) = \widehat{F}_{z_0+z_{\alpha_0},b}(t \oplus t_n).$$

By Lemma 2.17, we have $\widehat{F}_{z_0+z_{\alpha_0},b}(t \oplus t_n) = \widehat{F}_{z_0+z_\alpha,b}(t)$, Our result now follows by substituting $z = \lambda(t)$ to the distribution relation of Proposition 2.15, noting that $F_{z_0,b}(\alpha z)|_{z=\lambda(t)} = F_{z_0,b}(z)|_{z=\lambda([\alpha]t)}$. □

Remark 3.8.
(1) The function F_1^{col} is characterized as the unique function of the form $F_1^{\mathrm{col}} = \int dF_1$ satisfying the distribution relation.
(2) The convergence property for $\widehat{F}_{z_0,1}(t)$ shows that the function F_1^{col} converges on any point in $U(\mathbb{C}_p)$.
(3) Furthermore, the expansion of $\widehat{F}_b^{\mathrm{col}}(t)$ for $b > 1$ shows that F_b^{col} is defined on $E(\mathbb{C}_p)$.

3.3. **p-adic Eisenstein-Kronecker function.** We will now define the p-adic Eisenstein-Kronecker function. We will define this function to be the iterated Coleman integral of F_b^{col} whose constant term is chosen to satisfy the distribution relation. We first prepare a proposition concerning integration and the distribution relation.

Proposition 3.9. *Let m and b be integers ≥ 0. Suppose $E_{m,b}^{\mathrm{col}}$ is a Coleman function on U which satisfies the distribution relation*

$$\sum_{z_\alpha \in E[\alpha]} E_{m,b}^{\mathrm{col}}(z + z_\alpha) = \alpha^{1-m}\overline{\alpha}^{1-b} E_{m,b}^{\mathrm{col}}(\alpha z). \tag{7}$$

for any non-zero $\alpha \in \mathcal{O}_{\boldsymbol{K}}$. Then there exists a unique integration $E_{m+1,b}^{\mathrm{col}} := -\int E_{m,b}^{\mathrm{col}}\omega$ of $-E_{m,b}^{\mathrm{col}}\omega$ satisfying the distribution relation

$$\sum_{z_\alpha \in E[\alpha]} E_{m+1,b}^{\mathrm{col}}(z + z_\alpha) = \alpha^{-m}\overline{\alpha}^{1-b} E_{m+1,b}^{\mathrm{col}}(\alpha z) \tag{8}$$

for any non-zero $\alpha \in \mathcal{O}_{\boldsymbol{K}}$.

Proof. Let $\widetilde{E}_{m+1,b} := -\int E_{m,b}^{\mathrm{col}}\omega$ be any Coleman integral of $-E_{m,b}^{\mathrm{col}}\omega$. For any non-zero $\alpha \in \mathcal{O}_{\boldsymbol{K}}$, let

$$c_\alpha := \sum_{z_\alpha \in E[\alpha]} \widetilde{E}_{m+1,b}(z + z_\alpha) - \alpha^{-m}\overline{\alpha}^{1-b}\widetilde{E}_{m+1,b}(\alpha z).$$

Then the relation (7) shows that $dc_\alpha = 0$, hence the property of Coleman integration shows that c_α is a constant in $\mathbb{C}_p$. For any non-zero α, $\beta \in \mathcal{O}_{\boldsymbol{K}}$, we have

$$\begin{aligned}
\sum_{z_{\alpha\beta} \in E[\alpha\beta]} \widetilde{E}_{m+1,b}(z + z_{\alpha\beta}) &= \sum_{\substack{z_\alpha \in E[\alpha], \\ \widetilde{z}_\beta \in E[\alpha\beta]/E[\alpha]}} \widetilde{E}_{m+1,b}(z + z_\alpha + \widetilde{z}_\beta) \\
&= \sum_{\widetilde{z}_\beta \in E[\alpha\beta]/E[\alpha]} \Big(c_\alpha + \alpha^{-m}\overline{\alpha}^{1-b}\widetilde{E}_{m+1,b}(\alpha z + \alpha\widetilde{z}_\beta)\Big) \\
&= N(\beta)c_\alpha + \alpha^{-m}\overline{\alpha}^{1-b}c_\beta + (\alpha\beta)^{-m}(\overline{\alpha\beta})^{1-b}\widetilde{E}_{m+1,b}(\alpha\beta z),
\end{aligned}$$

where the last equality follows from the definition of c_β and the fact that we have an isomorphism $E[\alpha\beta]/E[\alpha] \cong E[\beta]$ given by $\widetilde{z}_\beta \mapsto z_\beta := \alpha\widetilde{z}_\beta$. By reversing the roles of α and β, we see from a similar calculation that the above is also equal to

$$N(\alpha)c_\beta + \beta^{-m}\overline{\beta}^{1-b}c_\alpha + (\alpha\beta)^{-m}(\overline{\alpha\beta})^{1-b}\widetilde{E}_{m+1,b}(\alpha\beta z).$$

This shows that we have $(N(\beta) - \beta^{-m}\overline{\beta}^{1-b})c_\alpha = (N(\alpha) - \alpha^{-m}\overline{\alpha}^{1-b})c_\beta$, hence the constant

$$c := c_\alpha/(N(\alpha) - \alpha^{-m}\overline{\alpha}^{1-b})$$

is independent of the choice of $\alpha \in \mathcal{O}_{\boldsymbol{K}}$. If we let

$$E_{m+1,b}^{\mathrm{col}}(z) := \widetilde{E}_{m+1,b}(z) - c,$$

then this function satisfies (8) for any non-zero $\alpha \in \mathcal{O}_{\boldsymbol{K}}$. $\square$

Definition 3.10. For integers m, b with $b \geq 0$, we define the *p-adic Eisenstein-Kronecker series* $E^{\mathrm{col}}_{m,b}$ recursively going up and down as follows. We first let $E^{\mathrm{col}}_{0,b} := (-1)^{b-1} F^{\mathrm{col}}_b$. Then by Proposition 3.7, this function satisfied the distribution relation (7). For $m > 0$, we let $E^{\mathrm{col}}_{m,b}$ be the Coleman function recursively defined by $E^{\mathrm{col}}_{m,b} = -\int E^{\mathrm{col}}_{m-1,b}\omega$, with the constant term normalized as in Proposition 3.9 to satisfy the distribution relation

$$\sum_{z_\alpha \in E[\alpha]} E^{\mathrm{col}}_{m,b}(z + z_\alpha) = \alpha^{1-m}\overline{\alpha}^{1-b} E^{\mathrm{col}}_{m,b}(\alpha z). \tag{9}$$

For $m < 0$, we define $E^{\mathrm{col}}_{m,b}$ recursively by the formula $dE^{\mathrm{col}}_{m+1,b} = -E^{\mathrm{col}}_{m,b}\omega$.

Proposition 3.9 insures that such a choice of constant term when $m > 0$ is possible. Again, the convergence property of F_1 in Proposition 2.16 insures that $E^{\mathrm{col}}_{m,b}$ is defined on any point in $U(\mathbb{C}_p)$ if $b = 1$ and on $E(\mathbb{C}_p)$ if $b > 1$. When $b = 0$, note that $E^{\mathrm{col}}_{0,0} \equiv 1$. This shows that we have $E^{\mathrm{col}}_{a,0} = 0$ for $a < 0$.

The reason we view $E^{\mathrm{col}}_{m,b}$ as a p-adic analogue of Eisenstein-Kronecker function is that this function interpolates values of the classical Eisenstein-Kronecker function at torsion points for $m \leq 0$ as follows.

Proposition 3.11. *Let a, b be integers ≥ 0. Then for any torsion point $z \in E(\overline{\mathbb{Q}})_{\mathrm{tor}}$ such that $z \neq 0$ if $b = 1$, we have*

$$E^{\mathrm{col}}_{-a,b}(z) = E_{-a,b}(z)/A^a =: e^*_{a,b}(z)/A^a.$$

Proof. Any torsion point z is of the form $z = z_0 + z_n$, where z_0 is a Teichmüller representative and z_n is a π^n-torsion point. Our result follows from the fact that

$$E^{\mathrm{col}}_{-a,b}(z) = (-1)^{a+b-1}\partial^a_{t,\log}\widehat{F}_{z_0,b}(t)|_{t=t_n}$$

and Corollary 2.18. $\square$

We will use the functions $E^{\mathrm{col}}_{m,b}(z)$ for $m > 0$ to define the p-adic Eisenstein-Kronecker number. We first prove that the values of $E^{\mathrm{col}}_{m,b}(z)$ are independent from the choice of the branch of the p-adic logarithm.

Lemma 3.12. *Let m and b be integers ≥ 0. Suppose that z is a point in $E(\mathbb{C}_p)\backslash]0[$. Then the value $E^{\mathrm{col}}_{m,b}(z)$ is independent from the choice of the branch of the p-adic logarithm.*

Proof. The proof is given by induction on m. The statement for $m = 0$ and $b = 1$ follows from the construction of F^{col}_1, which is independent of the choice of the branch of the p-adic logarithm. The statement for $m = 0$ and general b follows from the definition (6) of F^{col}_b, noting that L_n are rational functions. Suppose now that the statement is true for some $m \geq 0$. Consider the Coleman integral $\widetilde{E}_{m+1,b} = -\int E_{m,b}\omega$ as in Proposition 3.9. By Besser's

formalism [Bes3], Coleman integrals over points of good reduction (in other words, points in the smooth subscheme $U \subset X$ using the notations of §3.1) are free from the choice of the branch of the p-adic logarithm. Hence for z in $E(\mathbb{C}_p)\backslash]0[$, the value $\widetilde{E}_{m+1,b}(z)$ is independent of the choice of the branch. Furthermore, for a point in z in $E(\mathbb{C}_p)\backslash]0[$, the values $\widetilde{E}_{m+1,b}(z)$ and $\widetilde{E}_{m+1,b}(z+z_1)$ for $z_1 \in E[\mathfrak{p}]$ are also free from the choice of the branch. Hence the global constant $c = c_\pi$ of Proposition 3.9 is independent of the choice of the branch. This gives the statement for $E^{\mathrm{col}}_{m,b}(z)$. □

Remark 3.13. The restriction of $E^{\mathrm{col}}_{m,b}(z)$ to the residue disc $]0[$ is of the form

$$E^{\mathrm{col}}_{m,b}|_{]0[} = a_0(t) + a_1(t)\log t + a_2(t)(\log t)^2 + \cdots + a_n(t)(\log t)^n$$

for some n and $a_i(t) \in A(]0[)$, where the $a_i(t)$ are rigid analytic functions on $]0[$ free from the choice of the branch. Since $E^{\mathrm{col}}_{m,b}(z)$ is analytic on $E(\mathbb{C}_p)$, we see that $n = 0$ if $b \neq 1$. Therefore if $b \neq 1$, then the values $E^{\mathrm{col}}_{m,b}(z)$ for $z \in E(\mathbb{C}_p)$ is independent of the choice of the branch.

We define the p-adic Eisenstein-Kronecker number using this function.

Definition 3.14. For integers a, b with $b \geq 0$ and $z \in E(\mathbb{C}_p)$, with $z \neq 0$ if $b = 1$, we define the *p-adic Eisenstein-Kronecker number* $e^{\mathrm{col}}_{a,b}(z)$ by

$$e^{\mathrm{col}}_{a,b}(z) := E^{\mathrm{col}}_{-a,b}(z).$$

Note that $e^{\mathrm{col}}_{a,b}(z)$ is independent of the choice of the branch of the logarithm if $b \neq 1$ or $z \in E(\mathbb{C}_p)\backslash]0[$. The above terminology is justified by the following result, which follows directly from Proposition 3.11 and the definition of $e^{\mathrm{col}}_{a,b}(z)$.

Proposition 3.15. *Let a, b be integers such that a, $b \geq 0$, and suppose $z \in E(\overline{\mathbb{Q}})_{\mathrm{tor}}$ is such that $z \neq 0$ if $b = 1$. Then we have*

$$e^{\mathrm{col}}_{a,b}(z) = e^*_{a,b}(z)/A^a,$$

where $e^*_{a,b}(z)$ are the usual Eisenstein-Kronecker numbers.

We define a variant of the p-adic Eisenstein-Kronecker function as follows.

Definition 3.16. For any integers m and b such that $b \geq 0$, we let

$$E^{(p)}_{m,b}(z) := E^{\mathrm{col}}_{m,b}(z) - \frac{1}{\pi^m \overline{\pi}^b} E^{\mathrm{col}}_{m,b}(\pi z). \tag{10}$$

Remark 3.17. The function $E^{(p)}_{m,b}(z)$ on each residue disc is given as

$$E^{(p)}_{m,b}|_{]z_0[} = (-1)^{b-1}\widehat{E}^{(p)}_{z_0,m,b}(t),$$

where $\widehat{E}^{(p)}_{z_0,m,b}(t)$ is the power series defined in [BKT1] Definition 5.8. This fact is proved by induction on m, noting that both sides satisfy the distribution relation.

We next prove that the action of the Frobenius on the values of $E^{\mathrm{col}}_{m,b}$ at torsion points of order prime to $\mathfrak{p}$ may be described using the function $E^{(p)}_{m,b}(z)$. We denote by K_0 the maximal unramified extension of K, and let σ be a power of the absolute Frobenius of K_0 such that the fixed field of σ is $\boldsymbol{K}_{\mathfrak{p}}$. We denote again by σ any automorphism of $\mathbb{C}_p$ extending σ. Then we have the following.

Proposition 3.18. *Suppose m, b be integers ≥ 0, and let z_0 be a non-zero torsion point of $E(\mathbb{C}_p)$ of order prime to $\mathfrak{p}$. Then we have*

$$\left(1 - \frac{\sigma}{\pi^m \overline{\pi}^b}\right) E^{\mathrm{col}}_{m,b}(z_0) = E^{(p)}_{m,b}(z_0).$$

Proof. By the theory of complex multiplication, the action of the Frobenius on z_0 is given by $\sigma(z_0) = \pi z_0$. Hence it is sufficient to prove that

$$\sigma(E^{\mathrm{col}}_{m,b}(z)) = E^{\mathrm{col}}_{m,b}(\sigma(z)). \tag{11}$$

for any $z \in E(\mathbb{C}_p)\backslash]0[$. We prove this statement by induction on $m \geq 0$. We first consider the case when $m = 0$ and $b = 1$. Then $E^{\mathrm{col}}_{0,1}(z) = F^{\mathrm{col}}_1(z)$. Note that $\omega^* := dF^{\mathrm{col}}_1$ is a differential of the second kind defined over $\boldsymbol{K}$, hence we have $\sigma(\omega^*) = \omega^*$. This fact and the compatibility of Coleman integration with the action of a continuous endomorphism of $\mathbb{C}_p$ ([Col2] Corollary 2.1e) shows that the function

$$\sigma(F^{\mathrm{col}}_1(z)) - F^{\mathrm{col}}_1(\sigma(z))$$

is a constant. The distribution relation for F^{col}_1 proves that this constant is in fact zero. The case for $m = 0$ and $b > 1$ follows from (6), noting that $E^{\mathrm{col}}_{0,b}(z) := F^{\mathrm{col}}_b(z)$ and that L_n are rational functions defined over $\boldsymbol{K}_{\mathfrak{p}}$. Suppose next that (11) is true for some $m \geq 0$. Then the compatibility of Coleman integration with σ again shows that the function

$$\sigma(E^{\mathrm{col}}_{m+1,b}(z)) - E^{\mathrm{col}}_{m+1,b}(\sigma(z))$$

is a constant. The distribution relation (7) shows that this constant is in fact zero. Our assertion now follows by induction on m. □

3.4. **p-adic polylogarithm function.** We define the Coleman p-adic elliptic polylogarithm function using the p-adic Eisenstein-Kronecker series as follows.

Definition 3.19. For integers $m, n \geq 0$, we define the *Coleman p-adic elliptic polylogarithm functions* by

$$D^{\mathrm{col}}_{m,n} := (-1)^{n-1} \sum_{b=0}^{n} \frac{(F^{\mathrm{col}}_1)^{n-b}}{(n-b)!} E^{\mathrm{col}}_{m,b}.$$

Remark 3.20. We may express $E^{\mathrm{col}}_{m,b}$ in terms of $D^{\mathrm{col}}_{m,n}$ as

$$E^{\mathrm{col}}_{m,b} := (-1)^{b-1} \sum_{n=0}^{b} \frac{(F^{\mathrm{col}}_1)^{b-n}}{(b-n)!} D^{\mathrm{col}}_{m,n}.$$

Then the distribution relation (9) gives a formula which we may regard as the distribution relation for $D^{\mathrm{col}}_{m,n}$.

From the definition, we have $F_1(z) = \zeta(z) + e^*_{0,2} z$, hence $dF_1 = -\eta + e^*_{0,2}\omega$ for $\eta = \wp(z)dz = xdx/y$ is an algebraic differential. If we let $\omega^* := dF_1 = dF^{\mathrm{col}}_1$, then the Coleman p-adic elliptic polylogarithm functions satisfy by definition the differential equation

$$dD^{\mathrm{col}}_{m+1,n} = -D^{\mathrm{col}}_{m,n}\omega - D^{\mathrm{col}}_{m+1,n-1}\omega^*$$

for $m, n \geq 0$, where $D^{\mathrm{col}}_{m,n} := 0$ for $n < 0$ and $D^{\mathrm{col}}_{0,n} := L_n$.

In [BKT1], we defined the overconvergent p-adic elliptic polylogarithm functions $D^{(p)}_{m,n}$ as follows. Let $\Theta^{(p)}(z,w) := \Theta(z,w) - \overline{\pi}^{-1}\Theta(\pi z, \overline{\pi}^{-1}w)$ and

$$\Xi^{(p)}(z,w) := \exp(-F_1(z)w)\Theta^{(p)}(z,w).$$

We define the functions $L^{(p)}_n(z)$ as in Definition 2.10 by the expansion

$$\Xi^{(p)}(z,w) = \sum_{n\geq 0} L^{(p)}_n(z) w^{n-1}.$$

Then by [BKT1] Lemma 5.2, the function $L^{(p)}_n(z)$ is known to be a rational function on E defined over $\boldsymbol{K}$. In [BKT1] Theorem 6.14, we proved the following.

Theorem 3.21. *For integers m, n, there exists a unique system of overconvergent functions $D^{(p)}_{m,n}(z)$ on $\mathfrak{U}_K$ such that $D^{(p)}_{0,n} = L^{(p)}_{0,n}$, $D^{(p)}_{m,n} = 0$ if $n \leq 0$ and*

$$dD^{(p)}_{m,n} = -D^{(p)}_{m-1,n}\omega - D^{(p)}_{m,n-1}\omega^*$$

for $m, n > 0$.

Remark 3.22. By Remark 3.17 and [BKT1] Proposition 5.10, the function $D^{(p)}_{m,n}$ may be expressed in terms of $E^{(p)}_{m,b}(z)$ as

$$D^{(p)}_{m,n}(z) = (-1)^{n-1} \sum_{b=0}^{n} \frac{(F^{\mathrm{col}}_1)^{n-b}}{(n-b)!} E^{(p)}_{m,b}(z).$$

One may describe $D^{(p)}_{m,n}$ in terms of $D^{\mathrm{col}}_{m,n}$ using (10).

We next define a variant of the elliptic polylogarithm. In order to achieve this, we first regard $\lambda(t)$ as a Coleman function on E.

Lemma 3.23. *We let $\lambda(z)$ be a function on $E(\mathbb{C}_p)$ given on each residue disc $]z_0[$ as*

$$\lambda|_{]z_0[} := \lambda(t) \in A(]z_0[).$$

Then $\lambda(z)$ is a Coleman function on $E(\mathbb{C}_p)$.

Proof. By definition of $\lambda(z)$, we have $d\lambda = \omega$. Furthermore, since the formal logarithm is compatible with multiplication by $[\pi]$, we have

$$\left(1 - \frac{\phi^*}{\pi}\right)\lambda(z) = \lambda(z) - \frac{1}{\pi}\lambda(\pi z) \equiv 0.$$

Hence $\lambda(z)$ is a Coleman function by Lemma 3.3. □

Remark 3.24. If z is a non-zero torsion point in $E(\mathbb{C}_p)$, then $z = z_0 + z_n$ for some Teichmuller representative z_0 and π^n-torsion point z_n. If we denote by t_n the point on the unit disc corresponding to z_n, then we have $\lambda(z) = \lambda(t_n) = 0$. Hence the function $\lambda(z)$ is zero on any torsion point in $E(\mathbb{C}_p)$.

Definition 3.25. For integers m, n such that $m \geq 1$ and $n \geq 0$, we let

$$\Lambda_{m,n}(z) = \sum_{m_0=1}^{m} \sum_{n_0=0}^{n} \frac{\lambda(z)^{m-m_0} F_1^{\mathrm{col}}(z)^{n-n_0}}{(m-m_0)!(n-n_0)!} D^{\mathrm{col}}_{m_0,n_0}(z),$$

and $\Lambda_{0,n}(z) \equiv 0$. Furthermore, we let

$$\Lambda^*_{m,n}(z) = \Lambda_{m,n}(z) + \frac{\lambda(z)^m F_1^{\mathrm{col}}(z)^n}{m!n!}$$

for $m, n \geq 0$.

We will later use the above function to explicitly express the specialization of the elliptic polylogarithm to points on the elliptic curve.

4. Category of filtered overconvergent F-isocrystals

4.1. **Filtered overconvergent F-isocrystals.** Let K be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field k. We denote by K_0 the maximum unramified extension of $\mathbb{Q}_p$ in K, and by W its ring of integers. We denote by σ the lifting to W and K_0 of a certain fixed power of the absolute Frobenius of k.

Definition 4.1. We say that a pair $\mathscr{X} = (X, \overline{X})$ is a *smooth pair*, if X is a smooth scheme separated and of finite type over $\mathcal{O}_K$, and $\overline{X}$ is a smooth and proper compactification, separated and of finite type over $\mathcal{O}_K$, such that the complement $D = (\overline{X} \setminus X) \hookrightarrow \overline{X}$ is a normal crossing divisor of $\overline{X}$ relative to $\mathcal{O}_K$.

We will define the category of filtered overconvergent F-isocrystals $S(\mathscr{X})$ on a smooth pair $\mathscr{X}$. This category will be defined by pasting together the categories $S_{\mathrm{dR}}(\mathscr{X})$ and $S_{\mathrm{rig}}(\mathscr{X})$ over the category $S_{\mathrm{vec}}(\mathscr{X})$. We will first define the categories $S_{\mathrm{dR}}(\mathscr{X})$, $S_{\mathrm{rig}}(\mathscr{X})$ and $S_{\mathrm{vec}}(\mathscr{X})$. Let $\overline{X}_K := \overline{X} \otimes K$ and $X_K := X \otimes K$.

Definition 4.2. We define the category $S_{\mathrm{dR}}(\mathscr{X})$ to be the category consisting of objects the pair $M_{\mathrm{dR}} := (M_{\mathrm{dR}}, F^\bullet)$, where:

(1) M_{dR} is a coherent $\mathcal{O}_{\overline{X}_K}$-module with an integrable connection
$$\nabla_{\mathrm{dR}} : M_{\mathrm{dR}} \to M_{\mathrm{dR}} \otimes \Omega^1_{\overline{X}_K}(\log D_K)$$
with logarithmic poles along $D_K := D \otimes K$.

(2) $F^\bullet$ is a Hodge filtration, which is a descending exhaustive separated filtration on M_{dR} by coherent sub $\mathcal{O}_{\overline{X}_K}$-modules satisfying Griffiths transversality
$$\nabla_{\mathrm{dR}}(F^m M_{\mathrm{dR}}) \subset F^{m-1} M_{\mathrm{dR}} \otimes \Omega^1_{\overline{X}_K}(\log D_K).$$

Definition 4.3. We define the category $S_{\mathrm{rig}}(\mathscr{X})$ to be the category of overconvergent F-isocrystals $F\text{-Isoc}^\dagger(X_k/K_0)$ on the special fiber $X_k := X \otimes k$.

Denote by $\mathfrak{X}$ the formal completion of X with respect to the special fiber, and by $\mathfrak{X}_K$ and X_K^{an} the rigid analytic spaces over K associated to $\mathfrak{X}$ and X_K. We will use the same notations for $\overline{X}$.

Definition 4.4. We say that a set $V \subset \overline{X}_K^{\mathrm{an}}$ is a strict neighborhood of $\mathfrak{X}_K$ in X_K^{an}, if $V \cup (X_K^{\mathrm{an}}/\mathfrak{X}_K)$ is a covering of X_K^{an} for the Grothendieck topology.

For any abelian sheaf M on X_K^{an}, we let
$$j^\dagger M := \varinjlim_V \alpha_{V*}\alpha_V^* M,$$
where the limit is taken with respect to strict neighborhoods V of $\mathfrak{X}_K$ in X_K^{an} with inclusion $\alpha_V : V \hookrightarrow X_K^{\mathrm{an}}$. If M has a structure of a $\mathcal{O}_{X_K^{\mathrm{an}}}$-module, then $j^\dagger M$ has a structure of a $j^\dagger \mathcal{O}_{X_K^{\mathrm{an}}}$-module.

Definition 4.5. We denote by $S_{\mathrm{vec}}(\mathscr{X})$ the category of coherent $j^\dagger\mathcal{O}_{X_K^{\mathrm{an}}}$-modules with integrable connection.

We next define natural functors $\mathbf{F}_{\mathrm{dR}}$ and $\mathbf{F}_{\mathrm{rig}}$, which are exact functors from $S_{\mathrm{dR}}(\mathscr{X})$ and $S_{\mathrm{rig}}(\mathscr{X})$ to $S_{\mathrm{vec}}(\mathscr{X})$. Let $p_{\mathrm{dR}} : X_K^{\mathrm{an}} \to \overline{X}_K$ be the composition $X_K^{\mathrm{an}} \to X_K \hookrightarrow \overline{X}_K$.

Definition 4.6. We define the functor
$$\mathbf{F}_{\mathrm{dR}} : S_{\mathrm{dR}}(\mathscr{X}) \to S_{\mathrm{vec}}(\mathscr{X})$$
by associating to $M_{\mathrm{dR}} := (M_{\mathrm{dR}}, F^\bullet)$ the module $j^\dagger(p_{\mathrm{dR}}^* M_{\mathrm{dR}})$ with the connection induced from ∇_{dR}. The functor $\mathbf{F}_{\mathrm{dR}}$ is exact, since it is a composition of exact functors.

By [Ber1] 2.3.7, there exists a functor $\mathrm{Isoc}^\dagger(X_k/K_0) \to \mathrm{Isoc}^\dagger(X_k/K)$ defined by $M_{\mathrm{rig}} \to M_{\mathrm{rig}} \otimes K$. The objects of the category $\mathrm{Isoc}^\dagger(X_k/K)$ are realized by $j^\dagger\mathcal{O}_{X_K^{\mathrm{an}}}$-modules with integrable overconvergent connection, hence naturally gives an object in $S_{\mathrm{vec}}(\mathscr{X})$.

Definition 4.7. We define the functor
$$\mathbf{F}_{\mathrm{rig}} : S_{\mathrm{rig}}(\mathscr{X}) \to S_{\mathrm{vec}}(\mathscr{X})$$

by associating to $M_{\mathrm{rig}} := (M_{\mathrm{rig}}, \Phi)$ the $j^\dagger \mathcal{O}_{X_K^{\mathrm{an}}}$-module with integrable connection underlying the overconvergent isocrystal $M_{\mathrm{rig}} \otimes K$. The functor $\mathbf{F}_{\mathrm{rig}}$ is again exact.

Now we are ready to define the category $S_{\mathrm{syn}}(\mathscr{X})$.

Definition 4.8. We define $S_{\mathrm{syn}}(\mathscr{X})$ to be the category of *filtered overconvergent F-isocrystals*, whose objects are triples $\mathcal{M} := (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$, where

(1) M_{dR} is an object in $S_{\mathrm{dR}}(\mathscr{X})$, and M_{rig} is an object in $S_{\mathrm{rig}}(\mathscr{X})$.
(2) $\mathbf{p}$ is an isomorphism

$$\mathbf{p} : \mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}}) \xrightarrow{\cong} \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) \tag{12}$$

in $S_{\mathrm{vec}}(\mathscr{X})$.

A morphism $f : \mathcal{M} \to \mathcal{N}$ in this category is given by a pair $(f_{\mathrm{dR}}, f_{\mathrm{rig}})$ consisting of morphisms $f_{\mathrm{dR}} : M_{\mathrm{dR}} \to N_{\mathrm{dR}}$ and $f_{\mathrm{rig}} : M_{\mathrm{rig}} \to N_{\mathrm{rig}}$ in their respective categories, compatible with the comparison isomorphism $\mathbf{p}$.

Remark 4.9. Isomorphism (12) implies that the connection on $\mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}})$ is overconvergent. Hence $\mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}})$ and $\mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}})$ both represents objects in $\mathrm{Isoc}^\dagger(X_k/K)$.

If $X_k := \operatorname{Spec} k$, then the category $F\text{-}\mathrm{Isoc}^\dagger(X_k/K_0)$ is simply the category of finite K_0-vector spaces with a Frobenius structure. Hence for the smooth pair $\mathscr{V} := (\operatorname{Spec} \mathcal{O}_K, \operatorname{Spec} \mathcal{O}_K)$, then the category $S_{\mathrm{syn}}(\mathscr{V})$ consists of the triple $(M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$, where M_{dR} is a finite dimensional K-vector space with descending exhaustive separated filtration $F^\bullet$, M_{rig} is a finite dimensional K_0-vector space with a σ-linear Frobenius structure $\Phi : M_{\mathrm{rig}} \otimes_\sigma K_0 \cong M_{\mathrm{rig}}$, and $\mathbf{p}$ is an isomorphism $\mathbf{p} : M_{\mathrm{dR}} \cong M_{\mathrm{rig}} \otimes_{K_0} K$.

4.2. **Morphisms of smooth pairs.**

We next investigate the pull-back of filtered overconvergent F-isocrystals with respect to morphisms of smooth pairs.

Definition 4.10. We define a morphism $u : \mathscr{X} \to \mathscr{Y}$ of smooth pairs to be a morphism $u : \overline{X} \to \overline{Y}$ of schemes over $\operatorname{Spec} \mathcal{O}_K$ such that $u(X) \subset Y$.

If we let $u_{\mathrm{dR}} := u \otimes K$, then the pull back by $u_{\mathrm{dR}} : \overline{X}_K \to \overline{Y}_K$ induces a functor

$$u_{\mathrm{dR}}^* : S_{\mathrm{dR}}(\mathscr{Y}) \to S_{\mathrm{dR}}(\mathscr{X}).$$

Similarly, if we let $u_k := (u \otimes k)|_{X_k}$ be the restriction of u to the special fiber X_k, then the pull-back by $u_{\mathrm{rig}} : X_k \to Y_k$ induces a functor

$$u_{\mathrm{rig}}^* : S_{\mathrm{rig}}(\mathscr{Y}) \to S_{\mathrm{rig}}(\mathscr{X}).$$

These functors are compatible with the comparison isomorphism in the following sense.

Lemma 4.11. *Suppose we have a filtered overconvergent F-isocrystal $\mathcal{M} = (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$ in $S_{\mathrm{syn}}(\mathscr{Y})$. Then there exists a canonical isomorphism*

$$u^*(\mathbf{p}) : \mathbf{F}_{\mathrm{dR}}(u_{\mathrm{dR}}^* M_{\mathrm{dR}}) \xrightarrow{\cong} \mathbf{F}_{\mathrm{rig}}(u_{\mathrm{rig}}^* M_{\mathrm{rig}})$$

induced from the isomorphism $\mathbf{p} : \mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}}) \cong \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}})$.

Proof. Let u_{vec} be the morphism $u_K^{\mathrm{an}} : X_K^{\mathrm{an}} \to Y_K^{\mathrm{an}}$ induced from u. We denote by $u_{\mathrm{vec}}^* : S_{\mathrm{vec}}(\mathscr{Y}) \to S_{\mathrm{vec}}(\mathscr{X})$ the pull-back functor by u_{vec}. By [Ber1] Proposition 2.1.4, the functor $j^\dagger$ is compatible with the inverse image functor $u_K^{\mathrm{an}*}$. Hence there exists a canonical isomorphism

$$\mathbf{F}_{\mathrm{dR}}(u_{\mathrm{dR}}^* M_{\mathrm{dR}}) \xrightarrow{\cong} u_{\mathrm{vec}}^* \mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}}). \tag{13}$$

Similarly, taking $\otimes K$ is compatible with the pull-back morphism of overconvergent isocrystals. Hence there exists a canonical isomorphism

$$u_{\mathrm{vec}}^* \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) \xrightarrow{\cong} \mathbf{F}_{\mathrm{rig}}(u_{\mathrm{rig}}^* M_{\mathrm{rig}}). \tag{14}$$

The isomorphism $u^*(\mathbf{p})$ is defined as the composition of the above two maps and $u_{\mathrm{vec}}^*(\mathbf{p}) : u_{\mathrm{vec}}^* \mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}}) \cong u_{\mathrm{vec}}^* \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}})$. □

Remark 4.12. The compatibility with de Rham (13) is straightforward. The compatibility with rigid (14) exists due to general theory, but some work will be necessary to calculate it explicitly in terms of realizations.

We may now define the pull-back functor for filtered overconvergent F-isocrystals as follows.

Definition 4.13. Let $u : \mathscr{X} \to \mathscr{Y}$ be a morphism of smooth pairs. We define the functor $u_{\mathrm{syn}}^* : S_{\mathrm{syn}}(\mathscr{Y}) \to S_{\mathrm{syn}}(\mathscr{X})$ by associating to a a filtered overconvergent F-isocrystal $\mathcal{M} = (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$ in $S_{\mathrm{syn}}(\mathscr{Y})$ the triple

$$u_{\mathrm{syn}}^* \mathcal{M} = (u_{\mathrm{dR}}^* M_{\mathrm{dR}}, u_{\mathrm{rig}}^* M_{\mathrm{rig}}, u_{\mathrm{vec}}^*(\mathbf{p}))$$

in $S_{\mathrm{syn}}(\mathscr{X})$.

5. The p-adic elliptic polylogarithm

5.1. **Basics on the elliptic curve.** Let E be an elliptic curve as in §2.3. We let K be a finite extension of $\boldsymbol{K}_{\mathfrak{p}}$ in $\mathbb{C}_p$, and by abuse of notations, we again denote by E the extension of E to $\mathcal{O}_K$. If we let K_0 be the maximal unramified extension of $\mathbb{Q}_p$ in K, then we have $\boldsymbol{K}_{\mathfrak{p}} \subset K_0$ since p is a prime of good reduction. Denote by σ the power of the absolute Frobenius of K_0 such that the fixed field of σ is $\boldsymbol{K}_{\mathfrak{p}}$. We let E_0 be the extension to the ring of integers W of K_0 of the original E defined over $\mathcal{O}_{\boldsymbol{K}}$. The multiplication by $\phi := [\pi]$ gives a map $\phi : E_0 \to E_0$, which is a lifting of a Frobenius of E_0.

We let $\mathscr{E} = (E, E)$ be the smooth pair consisting of E over $\mathcal{O}_K$. We denote by $E_{K_0}^{\mathrm{an}}$ the rigid analytic space associated to $E_{K_0} := E_0 \otimes_W K_0$, and by E_K^{an} the rigid analytic space associated to $E_K := E \otimes_{\mathcal{O}_K} K$. An object M_{vec} in the category $\mathrm{Isoc}^\dagger(E_k/K)$ is realized as the category by a coherent $\mathcal{O}_{E_K^{\mathrm{an}}}$-module M with convergent integrable connection, and an object M_{rig} in the category

$\mathrm{Isoc}^\dagger(E_k/K_0)$ is realized by a coherent $\mathcal{O}_{E^{\mathrm{an}}_{K_0}}$-modules M_0 with convergent integrable connection. Let $\mathrm{pr} : E \to E_0$ be the natural projection. For a $\mathcal{O}_{E^{\mathrm{an}}_{K_0}}$-module M_0 with overconvergent integrable connection which realizes an object M_{rig} in $\mathrm{Isoc}^\dagger(E_k/K_0)$, the object $M_{\mathrm{rig}} \otimes K$ corresponding to the image by the functor

$$\mathrm{Isoc}^\dagger(E_k/K_0) \to \mathrm{Isoc}^\dagger(E_k/K)$$

is represented by the $\mathcal{O}_{E^{\mathrm{an}}_K}$-module with overconvergent integrable connection $\mathrm{pr}^* M_0$.

We let $H^1_{\mathrm{dR}}(E_{K_0}/K_0)$ be the de Rham cohomology of E_{K_0} over K_0. Then the invariant differential $\omega = dx/y$ defines a cohomology class $\underline{\omega}_{\mathrm{dR}}$ in $H^1_{\mathrm{dR}}(E_{K_0}/K_0)$, and since E has complex multiplication, the differential of the second kind $\omega^* := dF_1 = \eta + e^*_{0,2}\omega$ also defines a cohomology class $\underline{\omega}^*_{\mathrm{dR}}$ in $H^1_{\mathrm{dR}}(E_{K_0}/K_0)$. Then $\{\underline{\omega}_{\mathrm{dR}}, \underline{\omega}^*_{\mathrm{dR}}\}$ form a basis of $H^1_{\mathrm{dR}}(E_{K_0}/K_0)$, and the Hodge filtration on $H^1_{\mathrm{dR}}(E_{K_0}/K_0)$ is given by

$$F^0 H^1_{\mathrm{dR}} = H^1_{\mathrm{dR}}(E_{K_0}/K_0), \qquad F^1 H^1_{\mathrm{dR}} = K_0 \underline{\omega}_{\mathrm{dR}}, \qquad F^2 H^1_{\mathrm{dR}} = 0.$$

Next, let $H^1_{\mathrm{rig}}(E_k/K_0)$ be the rigid cohomology of E_k over K_0. Then GAGA gives a natural isomorphism

$$\mathbf{p} : H^1_{\mathrm{dR}}(E_{K_0}/K_0) \xrightarrow{\cong} H^1_{\mathrm{rig}}(E_k/K_0). \tag{15}$$

We denote by $\underline{\omega}_{\mathrm{rig}}$ and $\underline{\omega}^*_{\mathrm{rig}}$ the basis of $H^1_{\mathrm{rig}}(E_k/K_0)$ given by $\mathbf{p}(\underline{\omega}_{\mathrm{dR}}) = \underline{\omega}_{\mathrm{rig}}$ and $\mathbf{p}(\underline{\omega}^*_{\mathrm{dR}}) = \underline{\omega}^*_{\mathrm{rig}}$. Rigid cohomology has an action of the Frobenius $\Phi : H^1_{\mathrm{rig}}(E_k/K_0) \otimes_\sigma K_0 \to H^1_{\mathrm{rig}}(E_k/K_0)$, given on the basis $\underline{\omega}_{\mathrm{rig}}$ and $\underline{\omega}^*_{\mathrm{rig}}$ by

$$\Phi(\underline{\omega}) = \pi \underline{\omega}, \qquad\qquad \Phi(\underline{\omega}^*) = \overline{\pi}\underline{\omega}^*.$$

We let $H^1(\mathscr{E})$ be the filtered overconvergent F-isocrystal

$$H^1(\mathscr{E}) := (H^1_{\mathrm{dR}}(E_K/K), H^1_{\mathrm{rig}}(E_k/K_0), \mathbf{p})$$

in $S(\mathscr{V})$, where $H^1_{\mathrm{dR}}(E_K/K) := H^1_{\mathrm{dR}}(E_{K_0}/K_0) \otimes K$ is the de Rham cohomology of E_K over K and $\mathbf{p}$ is the isomorphism of (15) $\otimes K$.

We define $\mathscr{H} = (\mathscr{H}_{\mathrm{dR}}, \mathscr{H}_{\mathrm{rig}}, \mathbf{p})$ to be $\mathscr{H} := H^1(\mathscr{E})^\vee$. In other words, $\mathscr{H}$ is the filtered overconvergent F-isocrystal in $S(\mathscr{V})$ dual to $H^1(\mathscr{E})$. We denote by $\underline{\omega}^\vee_{\mathrm{dR}}$, $\underline{\omega}^{*\vee}_{\mathrm{dR}}$ the basis of $\mathscr{H}_{\mathrm{dR}}$ dual to $\underline{\omega}_{\mathrm{dR}}$, $\underline{\omega}^*_{\mathrm{dR}}$, and $\underline{\omega}^\vee_{\mathrm{rig}}$, $\underline{\omega}^{*\vee}_{\mathrm{rig}}$ the basis of $\mathscr{H}_{\mathrm{rig}}$ dual to $\underline{\omega}_{\mathrm{rig}}$, $\underline{\omega}^*_{\mathrm{rig}}$.

Let $u : \mathscr{E} \to \mathscr{V}$ be the structure morphism. By abuse of notation, we denote again by $\mathscr{H}$ the filtered overconvergent F-isocrystal $u^*\mathscr{H}$ in $S(\mathscr{E})$.

5.2. **Specialization on an elliptic curve.** We next define the specialization of a filtered overconvergent F-isocrystal on $\mathscr{E}$ to a point. Suppose $z \in E(\mathcal{O}_K)$, and let $i : \mathscr{V} \to \mathscr{E}$ be a map of smooth pairs defined by the map $i_z : \mathrm{Spec}\, \mathcal{O}_K \to E$ whose image is z. This induces a map $i_{\mathrm{rig}} : \mathrm{Spec}\, k \to E_k$ on the special fiber. There exists a unique lifting $i_{z_0} : \mathrm{Spec}\, W \to E_0$, which

we call the Teichmuller lifting, whose image is a torsion point z_0 of $E(W)$ of order prime to $\mathfrak{p}$. This map gives a diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O}_K & \xrightarrow{i_z} & E \\ \downarrow & & \downarrow \mathrm{pr} \\ \operatorname{Spec} W & \xrightarrow{i_{z_0}} & E_0, \end{array}$$

which itself is not necessarily commutative, but is commutative $\otimes k$. Hence we have a diagram of rigid analytic spaces

$$\begin{array}{ccc} \operatorname{Spm} K & \xrightarrow{i_z} & E_K^{\mathrm{an}} \\ \mathrm{pr}\downarrow & & \downarrow \mathrm{pr} \\ \operatorname{Spm} K_0 & \xrightarrow{i_{z_0}} & E_{K_0}^{\mathrm{an}}, \end{array} \tag{16}$$

which is again not necessarily commutative. The functor

$$i_{\mathrm{vec}}^* : \mathrm{Isoc}^\dagger(E_k/K) \to \mathrm{Isoc}^\dagger(\operatorname{Spec} k/K)$$

is realized as the pull back $M \mapsto i_z^* M$, and this realization is amenable with the pull back functor for de Rham cohomology. The functor

$$i_{\mathrm{rig}}^* : \mathrm{Isoc}^\dagger(E_k/K_0) \to \mathrm{Isoc}^\dagger(\operatorname{Spec} k/K_0)$$

is realized as the pull back $M_0 \mapsto i_{z_0}^* M_0$, and this realization is amenable with the Frobenius structure.

With the above consideration in mind, suppose M_0 is a coherent $\mathcal{O}_{E_{K_0}^{\mathrm{an}}}$-module with overconvergent integrable connection representing an object M_{rig} in $\mathrm{Isoc}^\dagger(E_k/K_0)$. Then the object $i_{\mathrm{vec}}^* \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}})$ and $\mathbf{F}_{\mathrm{rig}}(i_{\mathrm{rig}}^* M_{\mathrm{rig}})$ is represented by the modules

$$i_{\mathrm{vec}}^* \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) = (\mathrm{pr} \circ i_z)^* M_0, \qquad \mathbf{F}_{\mathrm{rig}}(i_{\mathrm{rig}}^* M_{\mathrm{rig}}) = (i_{z_0} \circ \mathrm{pr})^* M_0.$$

Since the diagram (16) is not commutative, the two objects above do not coincide. However, since i_z and i_{z_0} coincide $\otimes k$, we have the following.

Lemma 5.1. *There exists a canonical isomorphism*

$$\epsilon_{12} : (\mathrm{pr} \circ i_z)^* M_0 \xrightarrow{\cong} (i_{z_0} \circ \mathrm{pr})^* M_0$$

which realizes the isomorphism $i_{\mathrm{vec}}^* \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) \cong \mathbf{F}_{\mathrm{rig}}(i_{\mathrm{rig}}^* M_{\mathrm{rig}})$ *of* (14).

Proof. This follows from [Ber1] Proposition 2.2.17. □

The rest of this subsection is devoted to explicitly describing the isomorphism ϵ_{12} above, in terms of Taylor series as in [Ber1] (2.2.4). Let $\mathcal{E}$ be the formal completion of E with respect to the special fiber. Then the tubular neighborhood $]z_0[_{\mathcal{E}} \subset E_K^{\mathrm{an}}$ is isomorphic to the open unit disc $\mathbb{B}$. The local parameter $t = -2x/y$ of E at the origin gives by translation $\tau_{z_0} : z \mapsto z + z_0$

a local parameter again denoted by t in a neighborhood of z_0. Then t parameterizes the open unit disc $]z_0[_{\mathcal{E}}$. The restriction of $\Omega^1_{E_K^{\mathrm{an}}}$ to $]z_0[_{\mathcal{E}}$ is the free module $\mathcal{O}_{]z_0[_{\mathcal{E}}}dt$ generated by dt.

Suppose M is a coherent $\mathcal{O}_{E_K^{\mathrm{an}}}$-module with convergent connection ∇. We denote again by M the restriction of M to $]z_0[_{\mathcal{E}}$. Then the differential ∂_t dual to dt and the connection on M gives a differential $\nabla(\partial_t) : M \to M$ on $]z_0[_{\mathcal{E}}$. The maps $i_{z_0}, i_z : \operatorname{Spm} K \to E_K^{\mathrm{an}}$ factors through $]z_0[_{\mathcal{E}}$, and defines maps $i_{z_0}, i_z : \operatorname{Spm} K \to]z_0[_{\mathcal{E}}$. We let $\widetilde{t}$ be an element in K, such that i_{z_0} and i_z is given by $t \mapsto 0$ and $t \mapsto \widetilde{t}$. The isomorphism ϵ_{12} may now be constructed as follows.

Lemma 5.2. *Let the notations be as above. The isomorphism*

$$\epsilon_{12} : M \otimes_{i_z} K \to M \otimes_{i_{z_0}} K$$

is given explicitly by the expansion

$$\epsilon_{12}(m \otimes_{i_z} 1) = \sum_{k \geq 0} \frac{1}{k!} \nabla(\partial_t^k)(m) \otimes_{i_{z_0}} \widetilde{t}^k.$$

Proof. The expansion converges since the connection ∇ is convergent and $|\widetilde{t}| < 1$. Our assertion is [Ber1] (2.2.4). □

5.3. **The logarithm sheaf.** We now proceed to review the definition of the logarithm sheaf given in [BKT1] . The main result of this section is the explicit description of the specialization of the logarithm sheaf to a point on $U(\mathcal{O}_K)$, where $U := E \setminus [0]$.

The pair $\mathscr{U} := (U, E)$ is a smooth pair. The restriction of the logarithm sheaf $\mathscr{L}\mathrm{og}^N$ on $S(\mathscr{E})$ to $S(\mathscr{U})$ is given as follows.

Proposition 5.3. *The restriction of the elliptic logarithm sheaf to $\mathscr{U}$ is given as a system of objects $\mathscr{L}\mathrm{og}^N = (\mathscr{L}\mathrm{og}^N_{\mathrm{dR}}, \mathscr{L}\mathrm{og}^N_{\mathrm{rig}}, \mathbf{p})$ in $S_{\mathrm{syn}}(\mathscr{U})$ given as follows.*

(1) $\mathscr{L}\mathrm{og}^N_{\mathrm{dR}}$ *is the coherent $\mathcal{O}_{E_K}$-module*

$$\mathscr{L}\mathrm{og}^N_{\mathrm{dR}} = \bigoplus_{0 \leq m \leq k \leq N} \mathcal{O}_{E_K} \underline{\omega}^{m,k-m}_{\mathrm{dR}}$$

with basis $\underline{\omega}^{m,k-m}_{\mathrm{dR}}$, an integrable connection $\nabla(\underline{\omega}^{m,n}_{\mathrm{dR}}) = \underline{\omega}^{m+1,n}_{\mathrm{dR}} \otimes \omega + \underline{\omega}^{m,n+1}_{\mathrm{dR}} \otimes \omega^*$ and Hodge filtration

$$F^{-a} \mathscr{L}\mathrm{og}^N_{\mathrm{dR}} = \bigoplus_{0 \leq m \leq k \leq N, m \leq a} \mathcal{O}_{E_K} \underline{\omega}^{m,k-m}_{\mathrm{dR}}.$$

(2) $\mathscr{L}\mathrm{og}^N_{\mathrm{rig}}$ *is the overconvergent F-isocrystal which is realized by the coherent $j^\dagger \mathcal{O}_{U_{K_0}^{\mathrm{an}}}$-module $\mathscr{L}\mathrm{og}^N_{\mathrm{rig}} = \bigoplus_{0 \leq m \leq k \leq N} j^\dagger \mathcal{O}_{U_{K_0}^{\mathrm{an}}} \underline{\omega}^{m,k-m}_{\mathrm{rig}}$ with basis $\underline{\omega}^{m,k-m}_{\mathrm{rig}}$, an overconvergent integrable connection $\nabla(\underline{\omega}^{m,n}_{\mathrm{rig}}) =$*

$\underline{\omega}_{\mathrm{rig}}^{m+1,n} \otimes \omega + \underline{\omega}_{\mathrm{rig}}^{m,n+1} \otimes \omega^*$ *and Frobenius*

$$\Phi(\underline{\omega}_{\mathrm{rig}}^{m,n}) = \frac{1}{\pi^m \overline{\pi}^n} \sum_{k=n}^{N-m} \frac{(-F_1^{(p)}(z))^{k-n}}{(k-n)!} \underline{\omega}_{\mathrm{rig}}^{m,k}.$$

(3) *The comparison isomorphism* $\mathbf{p}$ *is given by* $\mathbf{p}(\underline{\omega}_{\mathrm{dR}}^{m,n}) = \underline{\omega}_{\mathrm{rig}}^{m,n} \otimes 1$.

The purpose of this section is to explicitly calculate the specialization $i_z^* \mathscr{L}\mathrm{og}^N$, for a point $i : \mathscr{V} \to \mathscr{U}$ given by the morphism $i_z : \operatorname{Spec} \mathcal{O}_K \to U$. Note that in this case, z_0 is a non-zero torsion point of order prime to $\mathfrak{p}$. The main statement will be given in Proposition 5.8. We first calculate the specializations of $\mathscr{L}\mathrm{og}_{\mathrm{dR}}^N$ and $\mathscr{L}\mathrm{og}_{\mathrm{rig}}^N$ with respect to i_z and i_{z_0}.

The module $i_z^* \mathscr{L}\mathrm{og}_{\mathrm{dR}}^N$ is given as the K-vector space

$$i_z^* \mathscr{L}\mathrm{og}_{\mathrm{dR}}^N = \bigoplus_{0 \leq m \leq k \leq N} K \underline{\omega}_{\mathrm{dR}}^{m,k-m}$$

with Hodge filtration given by subspaces

$$F^{-a} i_z^* \mathscr{L}\mathrm{og}_{\mathrm{dR}}^N = \bigoplus_{0 \leq m \leq k \leq N, m \leq a} K \underline{\omega}_{\mathrm{dR}}^{m,k-m} \subset i_z^* \mathscr{L}\mathrm{og}^N .$$

Similarly, the module $i_{z_0}^* \mathscr{L}\mathrm{og}_{\mathrm{rig}}^N$ is given by the K_0-vector space $i_{z_0}^* \mathscr{L}\mathrm{og}_{\mathrm{rig}}^N = \bigoplus_{0 \leq m \leq k \leq N} K_0 \underline{\omega}_{\mathrm{rig}}^{m,k-m}$, with Frobenius

$$\Phi(\underline{\omega}_{\mathrm{rig}}^{m,n}) = \frac{1}{\pi^m \overline{\pi}^n} \sum_{k=n}^{N-m} \frac{(-F_1^{(p)}(z_0))^{k-n}}{(k-n)!} \underline{\omega}_{\mathrm{rig}}^{m,k}.$$

The comparison morphism is given as follows. Let $\widetilde{F}_1(z) := F_1^{\mathrm{col}}(z) - F_1^{\mathrm{col}}(z_0)$, and let $\widetilde{u}_{m,n}(z) := \lambda(z)^m \widetilde{F}_1(z)^n / m! n!$ for $m, n \geq 0$. We take the convention that $\widetilde{u}_{m,n}(z) = 0$ if $m < 0$ or $n < 0$. The map $i_z^*(\mathbf{p})$ may be calculated as follows.

Proposition 5.4. *Let* $\mathbf{p}_z := i_z^*(\mathbf{p})$. *Then we have*

$$\mathbf{p}_z(\underline{\omega}_{\mathrm{dR}}^{m,n}) = \sum_{m_0=m}^{N} \sum_{n_0=n}^{N-m_0} \underline{\omega}_{\mathrm{rig}}^{m_0,n_0} \otimes \widetilde{u}_{m_0-m,n_0-n}(z). \tag{17}$$

Before we prove the proposition, we first prove the following lemma. We first define a series $a_{m,n}^{(0)}(z)$ of functions on $]z_0[_\varepsilon$ by $a_{0,0}^{(0)}(z) \equiv 1$ and $a_{m,n}^{(0)}(z) \equiv 0$ if $m \neq 0$ or $n \neq 0$. We then recursively define $a_{m,n}^{(k)}(z)$ by the formula

$$a_{m,n}^{(k+1)} = (\partial_t a_{m,n}^{(k)}) + a_{m-1,n}^{(k)} \partial_t \lambda + a_{m,n-1}^{(k)} \partial_t \widetilde{F}_1. \tag{18}$$

Lemma 5.5. *Let* k *be an integer* ≥ 0, *and let* $a_{m,n}^{(k)}(z)$ *be as above. Then we have*

$$(\partial_t^k \widetilde{u}_{m,n})(z_0) = a_{m,n}^{(k)}(z_0).$$

Proof. We first prove the equality

$$(\partial_t^k \widetilde{u}_{m,n})(z) = \sum_{m_0=0}^{m} \sum_{n_0=0}^{n} \widetilde{u}_{m-m_0,n-n_0}(z) a^{(k)}_{m_0,n_0}(z) \tag{19}$$

by induction on k. For $k = 0$, the formula is true by definition. Suppose (19) is true for $k \geq 0$. By taking the derivative ∂_t, we see that

$$\partial_t^{k+1}(\widetilde{u}_{m,n}) = \sum_{m_0=0}^{m} \sum_{n_0=0}^{n} (\widetilde{u}_{m-m_0,n-n_0} \partial_t a^{(k)}_{m_0,n_0} + a^{(k)}_{m_0,n_0} \widetilde{u}_{m-m_0-1,n-n_0} \partial_t \lambda + a^{(k)}_{m_0,n_0} \widetilde{u}_{m-m_0,n-n_0-1} \partial_t \widetilde{F}_1).$$

Hence (19) is true for $k+1$ by (18). Since $\widetilde{u}_{m,n}(z_0) = 1$ for $m = n = 0$ and $\widetilde{u}_{m,n}(z_0) = 0$ otherwise, our assertion follows by substituting $z = z_0$ into (19). □

We are now ready to prove Proposition 5.4.

Proof of Proposition 5.4. Let k be an integer ≥ 0, and let $a^{(k)}_{m,n}(z)$ be as above. We first prove by induction on k that

$$\nabla(\partial_t^k)(\underline{\omega}^{m,n}_{\mathrm{rig}}) = \sum_{m_0=m}^{N} \sum_{n_0=n}^{N-m_0} a^{(k)}_{m_0-m,n_0-n}(z) \underline{\omega}^{m_0,n_0}_{\mathrm{rig}}. \tag{20}$$

on $]z_0[_\mathcal{E}$. When $k = 0$, the statement is true from the definition of $a^{(0)}_{m,n}$. Suppose the statement is true for $k \geq 0$. If we differentiate (20) by $\nabla(\partial_t)$, we have

$$\nabla(\partial_t^{k+1})(\underline{\omega}^{m,n}_{\mathrm{rig}}) = \sum_{m_0=m}^{N} \sum_{n_0=n}^{N-m_0} ((\partial_t a^{(k)}_{m_0-m,n_0-n}) \underline{\omega}^{m_0,n_0}_{\mathrm{rig}} + a^{(k)}_{m_0-m,n_0-n}((\partial_t \lambda) \underline{\omega}^{m_0+1,n_0}_{\mathrm{rig}} + (\partial_t \widetilde{F}_1) \underline{\omega}^{m_0,n_0+1}_{\mathrm{rig}})).$$

Then by (18), we have the statement for $k + 1$. By Lemma 5.5, we have $(\partial_t^k \widetilde{u}_{m,n})(z_0) = a^{(k)}_{m,n}(z_0)$. Therefore, by substituting $z = z_0$ (which is $t = 0$ for the parameter t around z_0) to the definition of $a^{(k)}_{m,n}$, (20) gives

$$\nabla(\partial_t^k)(\underline{\omega}^{m,n}_{\mathrm{rig}}) \otimes 1 = \sum_{m_0=m}^{N} \sum_{n_0=n}^{N-m_0} \underline{\omega}^{m_0,n_0}_{\mathrm{rig}} \otimes (\partial_t^k \widetilde{u}_{m_0-m,n_0-n})(z_0).$$

By definition of $\mathbf{p}_z$, we have

$$\mathbf{p}_z(\underline{\omega}^{m,n}_{\mathrm{dR}}) := \sum_{k \geq 0} \frac{1}{k!} \nabla(\partial_t^k)(\underline{\omega}^{m,n}_{\mathrm{rig}}) \otimes \widetilde{t}^k.$$

Our assertion follows from the fact that $\sum_{k>0} \frac{1}{k!}(\partial_t^k \widetilde{u}_{m,n})(z_0) t^k$ is the Taylor expansion of $\widetilde{u}_{m,n}(z)$ at $t = 0$. □

We will next choose a different basis for $i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}}$ and $i_{z_0}^*\mathscr{L}\mathrm{og}^N_{\mathrm{rig}}$ which splits their respective structures. As in [BKT1] Definition 6.17, let

$$\underline{e}_{z_0}^{m,k} := \sum_{n=k}^{N-m} \frac{(-F_1^{\mathrm{col}}(z_0))^{n-k}}{(n-k)!}\underline{\omega}_{\mathrm{rig}}^{m,n}.$$

Then by [BKT1] Lemma 6.20, we have $\Phi(\underline{e}_{z_0}^{m,k}) = \pi^{-m}\overline{\pi}^{-k}\underline{e}_{z_0}^{m,k}$, hence the Frobenius on $i_{z_0}^*\mathscr{L}\mathrm{og}^N$ expressed using the basis $\{\underline{e}_{z_0}^{m,k}\}$ is trivialized. This gives in particular the isomorphism

$$i_{z_0}^*\mathscr{L}\mathrm{og}^N_{\mathrm{rig}} \xrightarrow{\cong} \prod_{w=0}^{N} \mathrm{Sym}^w \mathscr{H}_{\mathrm{rig}}, \qquad \underline{e}_{z_0}^{m,k} \mapsto \underline{\omega}_{\mathrm{rig}}^{\vee m}\underline{\omega}_{\mathrm{rig}}^{*\vee k}$$

compatible with the Frobenius structure. Similarly, if we let

$$\underline{e}_{z}^{m,k} := \sum_{n=k}^{N-m} \frac{(-F_1^{\mathrm{col}}(z))^{n-k}}{(n-k)!}\underline{\omega}_{\mathrm{dR}}^{m,n},$$

then the basis $\{\underline{e}_z^{m,k}\}$ of $i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}}$ gives an isomorphism

$$i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}} \xrightarrow{\cong} \prod_{w=0}^{N} \mathrm{Sym}^w \mathscr{H}_{\mathrm{dR}}, \qquad \underline{e}_{z}^{m,k} \mapsto \underline{\omega}_{\mathrm{dR}}^{\vee m}\underline{\omega}_{\mathrm{dR}}^{*\vee k}$$

compatible with the Hodge filtration. We now calculate the isomorphism $\mathbf{p}_z$ with respect to these basis.

Definition 5.6. For integers $m, n \geq 0$, we let

$$u_{m,n}(z) := \frac{\lambda(z)^m F_1^{\mathrm{col}}(z)^n}{m!n!}.$$

Note that this function is a p-adic analogue of the function $z^m\overline{z}^n/m!n!$ in the complex case.

Corollary 5.7. *Using the basis $\underline{e}_z^{m,k}$ and $\underline{e}_{z_0}^{m,k}$ above, we have*

$$\mathbf{p}_z(\underline{e}_z^{m,k}) = \sum_{m_0=m}^{N} \underline{e}_{z_0}^{m_0,k} \otimes u_{m_0-m,0}(z). \tag{21}$$

Proof. We calculate the right hand side of (17) in terms of $\underline{e}_{z_0}^{m,n}$. The sum $\sum_{n_0=n}^{N-m_0} \underline{\omega}_{\mathrm{rig}}^{m_0,n_0} \otimes \widetilde{u}_{m_0-m,n_0-n}(z_0)$ for a fixed m_0 is equal to

$$\lambda(z)^{m-m_0} \sum_{n_0=n}^{N-m_0}\sum_{k=n_0}^{N-m_0} \underline{e}_{z_0}^{m_0,k} \otimes \frac{F_1^{\mathrm{col}}(z_0)^{k-n_0}\widetilde{F}_1(z)^{n_0-n}}{(k-n_0)!(n_0-n)!}$$
$$= \lambda(z)^{m-m_0} \sum_{k=n}^{N-m_0} \underline{e}_{z_0}^{m_0,k} \otimes \frac{F_1^{\mathrm{col}}(z)^{k-n}}{(k-n)!}.$$

Our assertion now follows from the definition of $\underline{e}_z^{m,k}$ in the left hand side of (21) and the above equality. $\square$

Now we are ready to prove the main result of this subsection.

Proposition 5.8. *The restriction $i_z^*\mathscr{L}\mathrm{og}^N$ to a point $i_z : \mathscr{V} \to \mathscr{U}$ is given by the triple $(i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}}, i_{z_0}^*\mathscr{L}\mathrm{og}^N_{\mathrm{rig}}, \mathbf{p}_z)$, where*

(1) *$i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}}$ is the filtered K-vector space*

$$i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}} \cong \prod_{w=0}^{N} \mathrm{Sym}^w \mathscr{H}_{\mathrm{dR}}.$$

(2) *$i_{z_0}^*\mathscr{L}\mathrm{og}^N_{\mathrm{rig}}$ is the K_0-vector space with Frobenius structure*

$$i_{z_0}^*\mathscr{L}\mathrm{og}^N_{\mathrm{rig}} \cong \prod_{w=0}^{N} \mathrm{Sym}^w \mathscr{H}_{\mathrm{rig}}.$$

(3) *$\mathbf{p}_z$ is the isomorphism given in* (21).

As a corollary, we have the following.

Corollary 5.9 (Splitting principle)**.** *Let $i_z : \mathscr{V} \to \mathscr{U}$ be a morphism of smooth pairs, where $z \in U(\mathcal{O}_K)$ corresponds to a torsion point in $E(\mathbb{C}_p)_{\mathrm{tor}}$. Then we have an isomorphism*

$$i_z^*\mathscr{L}\mathrm{og}^N \cong \prod_{k=0}^{N} \mathrm{Sym}^k \mathscr{H}$$

in $S(\mathscr{V})$.

Proof. By the previous proposition, the only thing preventing $i_z^*\mathscr{L}\mathrm{og}^N$ from splitting is the comparison isomorphism $\mathbf{p}_z$. Our assertion now follows from the fact that $\lambda(z) = 0$ if z is a torsion point (see Remark 3.24). $\square$

5.4. **The polylogarithm sheaf.** We are now ready to calculate the specialization of the polylogarithm sheaf. The p-adic elliptic polylogarithm sheaf is a filtered overconvergent F-isocrystal on $\mathscr{U}$ given as an extension

$$0 \to \mathscr{L}\mathrm{og}^N(1) \to \mathscr{P}^N \to \mathscr{H} \to 0$$

characterized by a certain residue property at the boundary [0]. In this section, we will first recall the result of [BKT1] explicitly describing the polylogarithm sheaf. Then we will calculate its specialization to non-zero points of the elliptic curve.

We first define the differential forms $\boldsymbol{\omega}^\vee_{\mathrm{dR}}$, $\boldsymbol{\omega}^{*\vee}_{\mathrm{dR}}$ and $\boldsymbol{\omega}^\vee_{\mathrm{rig}}$, $\boldsymbol{\omega}^{*\vee}_{\mathrm{rig}}$.

Definition 5.10. We let

$$\boldsymbol{\omega}^\vee_{\mathrm{dR}} = -\underline{\omega}^{0,0}_{\mathrm{dR}} \otimes \omega^* + \sum_{n=1}^{N} L_n \underline{\omega}^{1,n-1}_{\mathrm{dR}} \otimes \omega, \qquad \boldsymbol{\omega}^{*\vee}_{\mathrm{dR}} = \sum_{n=0}^{N} L_n \underline{\omega}^{0,n}_{\mathrm{dR}} \otimes \omega.$$

be differential forms in $\Gamma(U_K, \mathscr{L}\mathrm{og}^N_{\mathrm{dR}} \otimes \Omega^1_{E_K})$, and

$$\boldsymbol{\omega}^\vee_{\mathrm{rig}} = -\underline{\omega}^{0,0}_{\mathrm{rig}} \otimes \omega^* + \sum_{n=1}^N L_n \underline{\omega}^{1,n-1}_{\mathrm{rig}} \otimes \omega, \qquad \boldsymbol{\omega}^{*\vee}_{\mathrm{rig}} = \sum_{n=0}^N L_n \underline{\omega}^{0,n}_{\mathrm{rig}} \otimes \omega$$

be differential forms in $\Gamma(U^{\mathrm{an}}_K, j^\dagger \mathscr{L}\mathrm{og}^N_{\mathrm{rig}} \otimes \Omega^1_{E_K})$.

We proved in [BKT1] Theorem 6.15 that the polylogarithm sheaf $\mathscr{P}^N$ is explicitly given in terms of the overconvergent functions $D^{(p)}_{m,n}$ of Theorem 3.21.

Theorem 5.11. *The filtered overconvergent F-isocrystal $\mathscr{P}^N$ is the triple $(\mathscr{P}^N_{\mathrm{dR}}, \mathscr{P}^N_{\mathrm{rig}}, \mathbf{p})$ given as follows.*

(1) *$\mathscr{P}^N_{\mathrm{dR}}$ is the coherent $\mathcal{O}_{U_K}$-module $\mathscr{P}^N_{\mathrm{dR}} := \mathscr{H}_{\mathrm{dR}} \bigoplus \mathscr{L}\mathrm{og}^N_{\mathrm{dR}}$, with connection $\nabla(\underline{\omega}^\vee_{\mathrm{dR}}) = \boldsymbol{\omega}^\vee_{\mathrm{dR}}$, $\nabla(\underline{\omega}^{*\vee}_{\mathrm{dR}}) = \boldsymbol{\omega}^{*\vee}_{\mathrm{dR}}$ and Hodge filtration*
$$F^m \mathscr{P}^N_{\mathrm{dR}} = F^m \mathscr{H}_{\mathrm{dR}} \bigoplus F^{m-1} \mathscr{L}\mathrm{og}^N_{\mathrm{dR}}.$$

(2) *$\mathscr{P}^N_{\mathrm{rig}}$ is the overconvergent F-isocrystal whose realization is given by the coherent $j^\dagger \mathcal{O}_{U^{\mathrm{an}}_{K_0}}$-module $\mathscr{P}^N_{\mathrm{rig}} := \mathscr{H}_{\mathrm{rig}} \bigoplus \mathscr{L}\mathrm{og}^N_{\mathrm{rig}}$, with connection $\nabla(\underline{\omega}^\vee_{\mathrm{rig}}) = \boldsymbol{\omega}^\vee_{\mathrm{rig}}$, $\nabla(\underline{\omega}^{*\vee}_{\mathrm{rig}}) = \boldsymbol{\omega}^{*\vee}_{\mathrm{rig}}$ and the Frobenius*
$$\Phi(\pi \underline{\omega}^\vee_{\mathrm{rig}}) = \underline{\omega}^\vee_{\mathrm{rig}} - \sum_{n=0}^N \frac{(-F^{(p)}_1)^{n+1}}{(n+1)!} \underline{\omega}^{0,n}_{\mathrm{rig}} + \sum_{m=1}^N \sum_{n=0}^{N-m} D^{(p)}_{m,n+1} \underline{\omega}^{m,n}_{\mathrm{rig}}$$
$$\Phi(\overline{\pi} \underline{\omega}^{*\vee}_{\mathrm{rig}}) = \underline{\omega}^{*\vee}_{\mathrm{rig}} + \sum_{m=0}^N \sum_{n=1}^{N-m} D^{(p)}_{m+1,n} \underline{\omega}^{m,n}_{\mathrm{rig}},$$
and the Frobenius $\frac{1}{p}\Phi$ on $\mathscr{L}\mathrm{og}^N_{\mathrm{rig}}$.

(3) *The comparison isomorphism $\mathbf{p}$ is given by $\mathbf{p}(\underline{\omega}^\vee_{\mathrm{dR}}) = \underline{\omega}^\vee_{\mathrm{rig}} \otimes 1$ and $\mathbf{p}(\underline{\omega}^{*\vee}_{\mathrm{dR}}) = \underline{\omega}^{*\vee}_{\mathrm{rig}} \otimes 1$.*

For integers m, n such that $m \geq 1$ and $n \geq 0$, we let
$$\widetilde{\Lambda}_{m,n}(z) := \sum_{m_0=1}^m \sum_{n_0=0}^n \widetilde{u}_{m-m_0,n-n_0}(z) D^{\mathrm{col}}_{m_0,n_0}(z),$$
and we define $\widetilde{\Lambda}_{0,n}(z) \equiv 0$. In addition, for integers $m, n \geq 0$, we let $\widetilde{\Lambda}^*_{m,n}(z) := \widetilde{\Lambda}_{m,n}(z) + \widetilde{u}_{m,n}(z)$. Then we have the following.

Proposition 5.12. *Let $\mathbf{p}_z := i^*_z(\mathbf{p})$. Then we have*
$$\mathbf{p}_z(\underline{\omega}^\vee_{\mathrm{dR}}) = \underline{\omega}^\vee_{\mathrm{rig}} \otimes 1 - \sum_{m=0}^N \sum_{n=0}^{N-m} \underline{\omega}^{m,n}_{\mathrm{rig}} \otimes (\widetilde{\Lambda}^*_{m,n+1}(z) - \widetilde{\Lambda}^*_{m,n+1}(z_0)).$$
$$\mathbf{p}_z(\underline{\omega}^{*\vee}_{\mathrm{dR}}) = \underline{\omega}^{*\vee}_{\mathrm{rig}} \otimes 1 - \sum_{m=0}^N \sum_{n=0}^{N-m} \underline{\omega}^{m,n}_{\mathrm{rig}} \otimes (\widetilde{\Lambda}_{m+1,n}(z) - \widetilde{\Lambda}_{m+1,n}(z_0)).$$

In order to prove this proposition, we first calculate the derivatives of $\widetilde{\Lambda}_{m,n}(z)$. We define $b^{(1)}_{m,n}(z)$ to be a function on $]z_0[_\varepsilon$ such that $b^{(1)}_{1,n}(z) = -D^{\mathrm{col}}_{0,n}(z)\partial_t\lambda(z)$ for $n \geq 0$, and $b^{(1)}_{m,n} \equiv 0$ otherwise. For $k > 0$, we recursively define the functions $b^{(k)}_{m,n}(z)$ by the formula

$$b^{(k+1)}_{m,n} = \partial_t b^{(k)}_{m,n} + b^{(k)}_{m-1,n}\partial_t\lambda + b^{(k)}_{m,n-1}\partial_t\widetilde{F}_1. \tag{22}$$

Then we have the following.

Lemma 5.13. *Let $k \geq 1$, and let $b^{(k)}_{m,n}(z)$ be as above. Then we have*

$$\partial_t^k(\widetilde{\Lambda}_{m,n}(z)) = \sum_{m_0=1}^{m}\sum_{n_0=0}^{n} \widetilde{u}_{m-m_0,n-n_0}(z) b^{(k)}_{m_0,n_0}(z). \tag{23}$$

Proof. We prove by induction on k. By definition and the differential property of $D^{\mathrm{col}}_{m,n}(z)$, we see that

$$\partial_t(\widetilde{\Lambda}_{m,n})(z) = -\sum_{n_0=0}^{n} \widetilde{u}_{m-1,n-n_0}(z) D^{\mathrm{col}}_{0,n_0}(z)\partial_t\lambda(z). \tag{24}$$

Hence the statement is true for $k = 1$. Suppose the statement is true for $k \geq 1$. By differentiating (23) by ∂_t, we have

$$\begin{aligned}\partial_t^{k+1}(\widetilde{\Lambda}_{m,n}(z)) = \sum_{m_0=1}^{m}\sum_{n_0=0}^{n} (&\widetilde{u}_{m-m_0,n-n_0}\partial_t b^{(k)}_{m_0,n_0} \\ &+ \widetilde{u}_{m-m_0-1,n-n_0} b^{(k)}_{m_0,n_0}\partial_t\lambda + \widetilde{u}_{m-m_0,n-n_0-1} b^{(k)}_{m_0,n_0}\partial_t\widetilde{F}_1).\end{aligned}$$

Then (22) gives our assertion. □

Lemma 5.14. *Let k be an integer ≥ 1, and let $b^{(k)}_{m,n}(z)$ be as above. Then we have*

$$\nabla(\partial_t^k)(\underline{\omega}^\vee_{\mathrm{rig}}) = -\sum_{m=0}^{N}\sum_{n=0}^{N-m} (b^{(k)}_{m,n+1}(z) + a^{(k)}_{m,n+1}(z))\underline{\omega}^{m,n}_{\mathrm{rig}}, \tag{25}$$

$$\nabla(\partial_t^k)(\underline{\omega}^{*\vee}_{\mathrm{rig}}) = -\sum_{m=0}^{N}\sum_{n=0}^{N-m} b^{(k)}_{m+1,n}(z)\underline{\omega}^{m,n}_{\mathrm{rig}}. \tag{26}$$

Proof. We prove by induction on k. Suppose $k = 1$. Then by definition, we have

$$\nabla(\partial_t)(\underline{\omega}^\vee_{\mathrm{rig}}) = -(\partial_t\widetilde{F}_1)\underline{\omega}^{0,0}_{\mathrm{rig}} + \sum_{n=0}^{N-1} L_{n+1}(\partial_t\lambda)\underline{\omega}^{1,n}_{\mathrm{rig}}$$

$$\nabla(\partial_t)(\underline{\omega}^{*\vee}_{\mathrm{rig}}) = \sum_{n=0}^{N} L_n(\partial_t\lambda)\underline{\omega}^{0,n}_{\mathrm{rig}}.$$

Hence the statement for $k=1$ follows from the definition of $b^{(k)}_{m,n}$ and the fact that $D^{\mathrm{col}}_{0,n}(z)=L_n(z)$. We next assume that (25) is true for some $k>0$. We have

$$\begin{aligned}\nabla(\partial_t^{k+1})(\underline{\omega}^{*\vee}_{\mathrm{rig}}) &= \nabla(\partial_t)\circ\nabla(\partial_t^k)(\underline{\omega}^{*\vee}_{\mathrm{rig}})\\ &= -\sum_{m=0}^{N}\sum_{n=0}^{N-m}(\partial_t b^{(k)}_{m,n+1})\underline{\omega}^{m,n}_{\mathrm{rig}} + b^{(k)}_{m,n+1}((\partial_t\lambda)\underline{\omega}^{m+1,n}_{\mathrm{rig}} + (\partial_t\widetilde{F}_1)\underline{\omega}^{m,n+1}_{\mathrm{rig}})\\ &\qquad + (\partial_t a^{(k)}_{m,n+1})\underline{\omega}^{m,n}_{\mathrm{rig}} + a^{(k)}_{m,n+1}((\partial_t\lambda)\underline{\omega}^{m+1,n}_{\mathrm{rig}} + (\partial_t\widetilde{F}_1)\underline{\omega}^{m,n+1}_{\mathrm{rig}}).\end{aligned}$$

Similarly, if we differentiate (26), then we have

$$\begin{aligned}\nabla(\partial_t^{k+1})(\underline{\omega}^{*\vee}_{\mathrm{rig}}) &= \nabla(\partial_t)\circ\nabla(\partial_t^k)(\underline{\omega}^{*\vee}_{\mathrm{rig}})\\ &= -\sum_{m=0}^{N}\sum_{n=0}^{N-m}(\partial_t b^{(k)}_{m+1,n})\underline{\omega}^{m,n}_{\mathrm{rig}} + b^{(k)}_{m+1,n}((\partial_t\lambda)\underline{\omega}^{m+1,n}_{\mathrm{rig}} + (\partial_t\widetilde{F}_1)\underline{\omega}^{m,n+1}_{\mathrm{rig}}).\end{aligned}$$

The assertion for $k+1$ follows from (18) and (22). $\square$

Proof of Proposition 5.12. Again, we have $\widetilde{u}_{m,n}(z_0)=0$. Then (23) shows that $(\partial_t^k\widetilde{\Lambda}_{m,n})(z_0)=b^{(k)}_{m,n}(z_0)$, and (19) shows that $(\partial_t^k\Lambda^*_{m,n})(z_0)=(b^{(k)}_{m,n}+a^{(k)}_{m,n})(z_0)$. Therefore, by substituting $z=z_0$, we have

$$\begin{aligned}\nabla(\partial_t^k)(\underline{\omega}^{\vee}_{\mathrm{rig}})\otimes 1 &= -\sum_{m=0}^{N}\sum_{n=0}^{N-m}\underline{\omega}^{m,n}_{\mathrm{rig}}\otimes(\partial_t^k\widetilde{\Lambda}^*_{m,n+1})(z_0)\\ \nabla(\partial_t^k)(\underline{\omega}^{*\vee}_{\mathrm{rig}})\otimes 1 &= -\sum_{m=0}^{N}\sum_{n=0}^{N-m}\underline{\omega}^{m,n}_{\mathrm{rig}}\otimes(\partial_t^k\widetilde{\Lambda}_{m+1,n})(z_0).\end{aligned}$$

Since $\sum_{k>0}\frac{1}{k!}(\partial_t^k\widetilde{\Lambda}_{m,n})(z_0)t^k$ and $\sum_{k>0}\frac{1}{k!}(\partial_t^k\widetilde{\Lambda}^*_{m,n})(z_0)t^k$ are the Taylor expansions of $\widetilde{\Lambda}_{m,n}(z)$ and $\widetilde{\Lambda}^*_{m,n}(z)$ at $z=z_0$ without the constant term, our assertion now follows from the definition of the isomorphism $\mathbf{p}_z$. $\square$

If we use the basis $\{\underline{e}^{m,b}_{z_0}\}$ of $i^*_{z_0}\mathscr{L}\mathrm{og}^N_{\mathrm{rig}}$, then the Frobenius structure on $i^*_{z_0}\mathscr{P}^N_{\mathrm{rig}}$ is given as follows.

Proposition 5.15. *The Frobenius on $i^*_{z_0}\mathscr{P}^N_{\mathrm{rig}}$ is expressed as*

$$\begin{aligned}\Phi(\pi\underline{\omega}^{\vee}_{\mathrm{rig}}) &= \underline{\omega}^{\vee}_{\mathrm{rig}} - \sum_{n=0}^{N}\left(1-\frac{\sigma}{\overline{\pi}^{n+1}}\right)\frac{F_1^{\mathrm{col}}(z_0)^{n+1}}{(n+1)!}\underline{e}^{0,n}_{z_0}\\ &\qquad + \sum_{m=1}^{N}\sum_{n=0}^{N-m}E^{(p)}_{m,n+1}(z_0)\underline{e}^{m,n}_{z_0},\\ \Phi(\overline{\pi}\underline{\omega}^{*\vee}_{\mathrm{rig}}) &= \underline{\omega}^{*\vee}_{\mathrm{rig}} + \sum_{m=0}^{N}\sum_{n=0}^{N-m}E^{(p)}_{m+1,n}(z_0)\underline{e}^{m,n}_{z_0},\end{aligned}$$

using the basis $\underline{e}_{z_0}^{m,n}$ *of* $i_{z_0}^* \mathscr{L}\mathrm{og}^N$.

If we take $\underline{\omega}_{z_0}^\vee$, $\underline{\omega}_{z_0}^{*\vee}$ to be

$$\underline{\omega}_{z_0}^\vee = \underline{\omega}_{\mathrm{rig}}^\vee - \sum_{n=0}^N \frac{F_1^{\mathrm{col}}(z_0)^{n+1}}{(n+1)!} \underline{e}_{z_0}^{0,n} + \sum_{m=1}^N \sum_{n=0}^{N-m} E_{m,n+1}^{\mathrm{col}}(z_0) \underline{e}_{z_0}^{m,n},$$
$$\underline{\omega}_{z_0}^{*\vee} = \underline{\omega}_{\mathrm{rig}}^{*\vee} + \sum_{m=0}^N \sum_{n=0}^{N-m} E_{m+1,n}^{\mathrm{col}}(z_0) \underline{e}_{z_0}^{m,n},$$

then we see that $\Phi(\underline{\omega}_{z_0}^\vee) = \pi^{-1} \underline{\omega}_{z_0}^\vee$ and $\Phi(\underline{\omega}_{z_0}^{*\vee}) = \overline{\pi}^{-1} \underline{\omega}_{z_0}^{*\vee}$. This shows that using this basis, we have an isomorphism

$$i_{z_0}^* \mathscr{P}_{\mathrm{rig}}^N \xrightarrow{\cong} \mathscr{H}_{\mathrm{rig}} \bigoplus i_{z_0}^* \mathscr{L}\mathrm{og}_{\mathrm{rig}}^N(1)$$

of Frobenius modules given by $\underline{\omega}_{z_0}^\vee \mapsto \underline{\omega}_{\mathrm{rig}}^\vee$ and $\underline{\omega}_{z_0}^{*\vee} \mapsto \underline{\omega}_{\mathrm{rig}}^{*\vee}$.

Recall from Definition 3.25 that for integers m, n such that $m \geq 1$ and $n \geq 0$, we let

$$\Lambda_{m,n}(z) := \sum_{m_0=1}^m \sum_{n_0=0}^n u_{m-m_0,n-n_0}(z) D_{m_0,n_0}^{\mathrm{col}}(z),$$

and $\Lambda_{0,n}(z) \equiv 0$. Furthermore, we let $\Lambda_{m,n}^*(z) = \Lambda_{m,n}(z) + u_{m,n}(z)$. Note that by definition, we have $\Lambda_{m,n}(z_0) = E_{m,n}^{\mathrm{col}}(z_0)$ for $m \geq 1$. Using the basis $\{\underline{e}_{z_0}^{m,n}\}$, we have the following.

Lemma 5.16. *We have*

$$\begin{aligned} \mathbf{p}_z(\underline{\omega}_{\mathrm{dR}}^\vee) &= \underline{\omega}_{z_0}^\vee \otimes 1 - \sum_{m=0}^N \sum_{n=0}^{N-m} \Lambda_{m,n+1}^*(z) \underline{e}_{z_0}^{m,n}, \\ \mathbf{p}_z(\underline{\omega}_{\mathrm{dR}}^{*\vee}) &= \underline{\omega}_{z_0}^{*\vee} \otimes 1 - \sum_{m=0}^N \sum_{n=0}^{N-m} \Lambda_{m+1,n}(z) \underline{e}_{z_0}^{m,n}. \end{aligned} \tag{27}$$

Proof. The second equality follows from the fact that

$$\sum_{k=0}^n \frac{F_1^{\mathrm{col}}(z_0)^{n-k}}{(n-k)!} \widetilde{\Lambda}_{m+1,k}(z) = \Lambda_{m+1,n}(z)$$

and $\Lambda_{m,n}(z_0) = E_{m,n}^{\mathrm{col}}(z_0)$. For the first equality, note that we have

$$\sum_{k=0}^n \frac{F_1^{\mathrm{col}}(z_0)^{n-k}}{(n-k)!} \widetilde{\Lambda}_{m,k+1}(z) = \Lambda_{m,n+1}(z) - \frac{F_1^{\mathrm{col}}(z_0)^{n+1}}{(n+1)!} D_{m,0}^{\mathrm{col}}(z)$$

for $m \geq 1$ and

$$\sum_{k=0}^n \frac{F_1^{\mathrm{col}}(z_0)^{n-k}}{(n-k)!} \widetilde{u}_{m,k+1}(z) = u_{m,n+1}(z) - \frac{F_1^{\mathrm{col}}(z_0)^{n+1}}{(n+1)!}$$

for $m \geq 0$. This implies that

$$\sum_{m=0}^{N}\sum_{n=0}^{N-m} \underline{\omega}_{\mathrm{rig}}^{m,n} \otimes \widetilde{\Lambda}^*_{m,n+1}(z) = \sum_{k=0}^{N-m} \underline{e}_{z_0}^{0,k} \otimes \left(u_{0,k+1}(z) - \frac{F_1(z_0)^{k+1}}{(k+1)!} \right)$$
$$+ \sum_{m=1}^{N}\sum_{k=0}^{N-m} \underline{e}_{z_0}^{m,k} \otimes \left(\Lambda^*_{m,k+1}(z) - \frac{F_1(z_0)^{k+1}}{(k+1)!} \sum_{m_0=0}^{m} \frac{\lambda(z)^{m-m_0}}{(m-m_0)!} D_{m_0,0(z)} \right).$$

Our assertion follows now from the definition of $\mathbf{p}_z$, $\underline{\omega}_{z_0}^{\vee}$, $\underline{\omega}_{z_0}^{\vee *}$, and the equality

$$\sum_{m_0=0}^{m} \frac{\lambda^{m-m_0}(z)}{(m-m_0)!} D^{\mathrm{col}}_{m_0,0}(z) - D^{\mathrm{col}}_{m,0}(z_0) = 0,$$

which may be seen from the fact that $f(z) = \sum_{m_0=0}^{m} \frac{\lambda^{m-m_0}(z)}{(m-m_0)!} D^{\mathrm{col}}_{m_0,0}(z)$ is a constant function since $f'(z) = 0$, and that $f(z_0) = D^{\mathrm{col}}_{m,0}(z_0)$. □

This gives the following theorem.

Theorem 5.17. *Suppose $i_z : \mathscr{V} \to \mathscr{U}$ is a morphism of smooth pairs. Then the specialization $i_z^*\mathscr{P}^N$ of the polylogarithm sheaf $\mathscr{P}^N$ on $S(\mathscr{U})$ is given by the triple $(i_z^*\mathscr{P}^N_{\mathrm{dR}}, i_{z_0}^*\mathscr{P}^N_{\mathrm{rig}}, \mathbf{p}_z)$, where*

(1) *$i_z^*\mathscr{P}^N_{\mathrm{dR}}$ is the filtered K-vector space*
$$i_z^*\mathscr{P}^N_{\mathrm{dR}} = \mathscr{H}_{\mathrm{dR}} \bigoplus i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}}(1).$$

(2) *$i_{z_0}^*\mathscr{P}^N_{\mathrm{rig}}$ is the K_0-vector space with Frobenius structure*
$$i_{z_0}^*\mathscr{P}^N_{\mathrm{rig}} = \mathscr{H}_{\mathrm{rig}} \bigoplus i_{z_0}^*\mathscr{L}\mathrm{og}^N_{\mathrm{rig}}(1).$$

(3) *The comparison isomorphism $\mathbf{p}_z$ is given by* (27).

5.5. **The specialization in cohomology.** The polylogarithm sheaf $\mathscr{P}^N$ is an extension of $\mathscr{H}$ by $\mathscr{L}\mathrm{og}^N(1)$ whose extension class in

$$\mathrm{Ext}^1_{S(\mathscr{U})}(\mathscr{H}, \mathscr{L}\mathrm{og}^N(1)) \cong H^1_{\mathrm{syn}}(\mathscr{U}, \mathscr{H}^\vee \otimes \mathscr{L}\mathrm{og}^N(1))$$

is the polylogarithm class. If we are given a point $i_z : \mathscr{V} \to \mathscr{U}$, then we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^1_{S(\mathscr{U})}(\mathscr{H}, \mathscr{L}\mathrm{og}^N(1)) & \xrightarrow{\cong} & H^1_{\mathrm{syn}}(\mathscr{U}, \mathscr{H}^\vee \otimes \mathscr{L}\mathrm{og}^N(1)) \\
{\scriptstyle i_z^*}\big\downarrow & & {\scriptstyle i_z^*}\big\downarrow \\
\mathrm{Ext}^1_{S(\mathscr{V})}(\mathscr{H}, i_z^*\mathscr{L}\mathrm{og}^N(1)) & \xrightarrow{\cong} & H^1_{\mathrm{syn}}(\mathscr{V}, \mathscr{H}^\vee \otimes i_z^*\mathscr{L}\mathrm{og}^N(1)),
\end{array}$$

and the pull back by i_z^* of the polylogarithm class corresponds to the extension $i_z^*\mathscr{P}^N \in \mathrm{Ext}^1_{S(\mathscr{V})}(\mathscr{H}, i_z^*\mathscr{L}\mathrm{og}^N(1))$.

Lemma 5.18. *Suppose $M = (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$ is an object in $S(\mathscr{V})$, such that $(1-\Phi)$ is surjective on M_{rig}. Then we have a canonical isomorphism*

$$\mathrm{Ext}^1_{S(\mathscr{V})}(K(0), M) \cong M_{\mathrm{dR}}/F^0 M_{\mathrm{dR}}.$$

Proof. This follows from the fact that for a module M, we have

$$\mathrm{Ext}^1_{S(\mathscr{V})}(K(0), M) = \mathrm{Coker}\left(F^0 M_{\mathrm{dR}} \bigoplus M_{\mathrm{rig}} \xrightarrow{\varphi} M_{\mathrm{dR}} \bigoplus M_{\mathrm{rig}}\right),$$

where $\varphi : (x, y) \mapsto (x - \mathbf{p}^{-1}(y), (1-\Phi)y)$. Since we have assumed that $(1-\Phi)$ is surjective, the above complex may be calculated by the natural inclusion $F^0 M_{\mathrm{dR}} \to M_{\mathrm{dR}}$. This gives our assertion. $\square$

In our case, the map $(1-\Phi)$ has no kernel on $\mathscr{H}^\vee \otimes i_z^* \mathscr{L}\mathrm{og}^N(1)$, hence it is surjective. The above lemma shows that we have in particular

$$\begin{aligned}(28)\quad \mathrm{Ext}^1_{S(\mathscr{V})}(\mathscr{H}, i_z^*\mathscr{L}\mathrm{og}^N(1)) &= \mathrm{Ext}^1_{S(\mathscr{V})}(K(0), \mathscr{H}^\vee \otimes i_z^*\mathscr{L}\mathrm{og}^N(1)) \\ &\cong \mathscr{H}^\vee_{\mathrm{dR}} \otimes i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}} / F^{-1}(\mathscr{H}^\vee_{\mathrm{dR}} \otimes i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}}).\end{aligned}$$

Theorem 5.19. *The specialization $i_z^*\mathscr{P}^N \in \mathrm{Ext}^1_{S(\mathscr{V})}(\mathscr{H}, \mathscr{L}\mathrm{og}^N(1))$ maps to the element*

$$\sum_{m=0}^{N}\sum_{n=0}^{N-m}\left(\Lambda^*_{m,n+1}(z)\underline{\omega}\otimes \underline{e}_z^{m,n} + \Lambda_{m+1,n}(z)\underline{\omega}^*\otimes \underline{e}_z^{m,n}\right)$$

in $\mathscr{H}^\vee_{\mathrm{dR}} \otimes i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}}$ through the isomorphism of (28).

Proof. This follows from the definition of the isomorphism given in Lemma 5.18 and Theorem 5.17. $\square$

Corollary 5.20. *Suppose z is a torsion point in $U(\mathcal{O}_K)$. Then the specialization $i_z^*\mathscr{P}^N \in \mathrm{Ext}^1_{S(\mathscr{V})}(\mathscr{H}, \mathscr{L}\mathrm{og}^N(1))$ maps to*

$$\sum_{m=1}^{N}\sum_{n=0}^{N-m}(-1)^n e^{\mathrm{col}}_{m,n+1}(z)\underline{\omega}\otimes\underline{e}_z^{m,n} + \sum_{m=0}^{N}\sum_{n=0}^{N-m}(-1)^{n-1}e^{\mathrm{col}}_{m+1,n}(z)\underline{\omega}^*\otimes\underline{e}_z^{m,n}.$$

Proof. From the fact that $\lambda(z) = 0$ (Remark 3.24) and the definitions of $\Lambda_{m,n}(z)$ and $E^{\mathrm{col}}_{m,n}(z)$, we see that $i_z^*\mathscr{P}^N$ maps to

$$\sum_{n=0}^{N-m}\frac{F_1^{\mathrm{col}}(z)^{n+1}}{(n+1)!}\underline{\omega}\otimes\underline{e}_z^{0,n} + \sum_{m=1}^{N}\sum_{n=0}^{N-m}(-1)^n E^{\mathrm{col}}_{m,n+1}(z)\underline{\omega}\otimes\underline{e}_z^{m,n}$$
$$+ \sum_{m=0}^{N}\sum_{n=0}^{N-m}(-1)^{n-1}E^{\mathrm{col}}_{m+1,n}(z)\underline{\omega}^*\otimes\underline{e}_z^{m,n}.$$

Our assertion follows from the definition of p-adic Eisenstein-Kronecker numbers given in Definition 3.14 and the fact that $\underline{\omega}\otimes\underline{e}_z^{0,n} \in F^{-1}(\mathscr{H}^\vee_{\mathrm{dR}} \otimes i_z^*\mathscr{L}\mathrm{og}^N_{\mathrm{dR}})$ for any $n \geq 0$. $\square$

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